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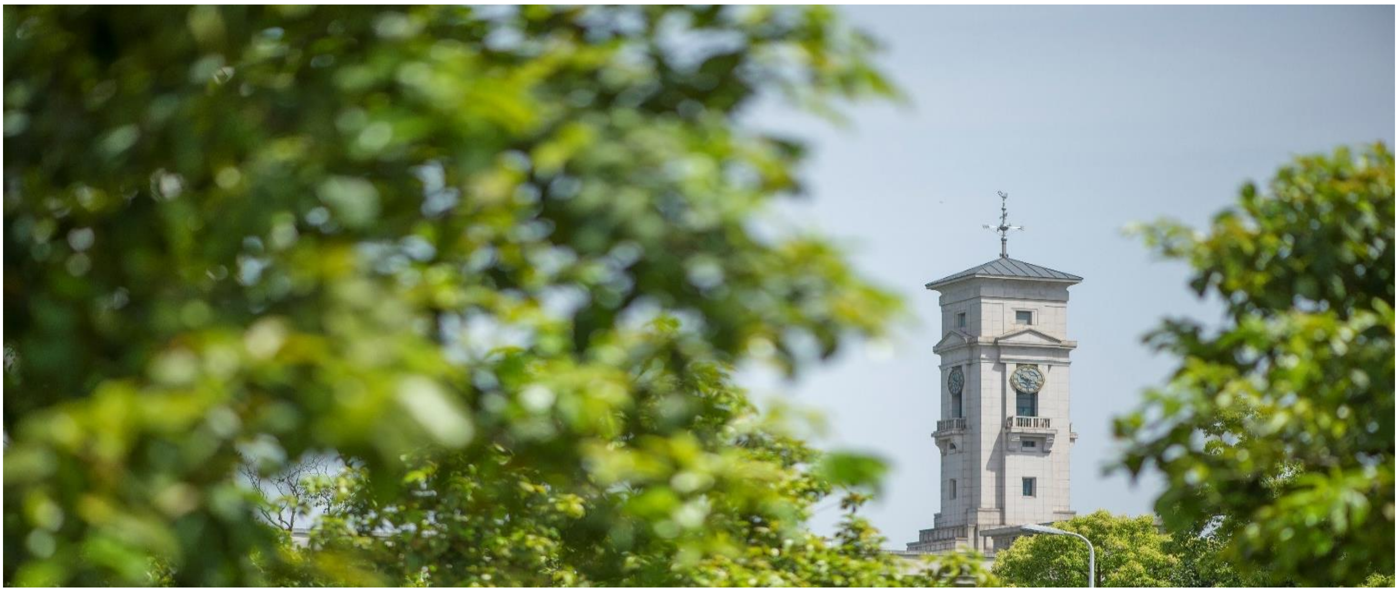

University of Nottingham Ningbo China, 199 Taikang East Road, Ningbo, 315100, China

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Overdetermined problems for p-Laplace and generalized Monge-Ampére equations

Behrouz Emamizadeh [*], Yichen Liu [†], Giovanni Porru[‡]

Abstract

We investigate overdetermined problems for p-Laplace and generalized Monge-Ampére equations. By using the theory of domain derivative we find duality results and a characterization of the overdetermined boundary conditions via minimization of suitable functionals with respect to the domain.

Key Words: Overdetermined problems, Domain derivative, Duality results, Domain functionals, p-Laplace equations, Generalized Monge-Ampére equations.

Mathematics Subject Classification: 35N25, 35A23, 35J96, 47J20, 52A40.

1 Introduction

Let D be a bounded smooth domain in $\mathbb{R}^N$. A point $x \in D$ will be denoted with $x = (x^1, \cdots, x^N)$. We also denote $u_i = \frac{\partial u}{\partial x^i}$, $u_{ij} = \frac{\partial^2 u}{\partial x^i \partial x^j}$, etc, the partial derivatives of u.

Let us recall the following well known overdetermined problem. Let c be a constant. If there exists a solution u to the Dirichlet problem

$$\Delta u = 1 \text{ in } D, \quad u = 0 \text{ on } \partial D \tag{1}$$

such that u satisfies the additional condition

$$|\nabla u| = c \text{ on } \partial D \tag{2}$$

[*]Behrouz.Emamizadeh@nottingham.edu.cn, School of Mathematical Sciences, The University of Nottingham Ningbo China

[†]yichen.liu07@yahoo.com, Department of Mathematical Sciences, Xi'an Jiaotong-Liverpool University, Suzhou, China

[‡]porru@unica.it, Department of Mathematics and Informatics, University of Cagliari, Italy

then D must be a ball. This result has been proved by J. Serrin [15] on 1971 using the moving plane method. At the same time, H. Weinberger [19] yields a different proof of the same result by using a Pohozaev identity and the maximum principle applied to a suitable P-function. The method of Weinberger requires less regularity of the boundary ∂D, but the method of Serrin can be easily applied to a large class of non-linear and fully non-linear operators. These two celebrated papers have inspired a great number of mathematicians, and the corresponding literature is nowadays very prominent. We refer to [2, 3, 5, 9, 11, 17] and references therein. For recent progress on this topic, we refer to the survey [10]. Among several ideas related to this overdetermined problem, we recall the following duality result [11].

Theorem 1.1 *Let $u \in C^2(D) \cap C^1(\bar{D})$ be a solution to Problem (1). The following statements are equivalent:*
(i) u satisfies condition (2).
(ii) The identity

$$\int_D v\,dx = c\int_{\partial D} v\,d\sigma \tag{3}$$

holds for all functions v harmonic in D.

Motivated by this result, we shall prove duality theorems for overdetermined problems involving p-Laplace equations as well as generalized Monge-Ampére equations. In case of generalized Monge-Ampére equations, the overdetermined boundary condition is not the same as (2), but condition (27) below. In the linear case ($\kappa = 1$) condition (27) reduces to the familiar condition $|\nabla u| = c$ on ∂D. If $1 < \kappa \leq N$, this condition involves ∇u as well as the second derivatives of u throughout the Newton tensor $T_{\kappa-1}(u)$. Furthermore, we consider suitable functionals of the domain D whose minimizers must satisfy the overdetermined boundary condition (2) for the p-Laplace problem, and condition (27) for the generalized Monge-Ampére problems. A crucial tool serving us shall be the domain derivative.

The paper is organized as follows. In Section 2 we introduce the notion of domain derivative. Some of our descriptions are formal, for a precise treatment of the domain derivative we refer to [16]. In particular, we find a sort of linearized equation of the p-Laplace equation $\Delta_p u = f(u)$ (see equation (10)), as well as a linearized equation of the generalized Monge-Ampére equation $S_\kappa(u) = f(u)$ (see equation (19)). These linearized equations are crucial to get our duality results. Sections 3 and 4 contain our main results. Section 3 is made of two subsections. In Subsection 3.1 we prove a duality result for a p-Laplace boundary value problem (see Theorem 3.1). In Subsection 3.2 we prove a duality result for a boundary value problem corresponding to a generalized Monge-Ampére equation (see Theorem 3.2). Also Section 4 is made of two subsections. In Section 4.1 we introduce a special functional associated with our p-Laplace equation in a domain D. We shall prove that the minimum of such functional with respect to D under the condition $|D| = \textit{constant}$ yields a condition for ∇u

on ∂D which is the same as used in Theorem 3.1 (i). In Section 4.2 we introduce a special functional associated with a generalized Monge-Ampére equation in a domain D. We shall prove that the minimum of such functional with respect to D under the condition $|D| = constant$ yields a condition for ∇u on ∂D which is the same as used in Theorem 3.2 (i).

2 Domain derivative

The theory of domain derivative is very useful in fields as shape optimization. From a mathematical point of view, it goes back to Hadamard [8] and Schiffer [14]. We recall shortly the definitions and refer to [16] for a careful treatment. If $\mathcal{L}(u)$ is a differential operator, we consider the Dirichlet problem:

$$\mathcal{L}(u) = f(u) \quad \text{in} \quad D, \quad u = 0 \quad \text{on} \quad \partial D, \tag{4}$$

where f is a smooth function such that problem (4) has a unique solution. Let I be the identity map. For a smooth (C^2 is enough) vector field $V : \mathbb{R}^N \to \mathbb{R}^N$, and $|t|$ small, define

$$D^t = (I + tV)(D).$$

Now, we consider the Dirichlet problem in D^t:

$$\mathcal{L}(u^t) = f(u^t) \quad \text{in} \quad D^t, \quad u^t = 0 \quad \text{on} \quad \partial D^t. \tag{5}$$

For $x \in D$ we define

$$v(x) = \lim_{t \to 0} \frac{u^t(x) - u(x)}{t}. \tag{6}$$

Clearly, since D^t depends on the vector field V, also v depends on V. By [16], v satisfies the boundary condition

$$v(x) = -\frac{\partial u}{\partial \nu} \; (V \cdot \nu) \quad \text{on} \quad \partial D, \tag{7}$$

where $\nu = (\nu^1, \cdots, \nu^N)$ is the unit exterior normal on ∂D.
To obtain the equation for v, we compute

$$\lim_{t \to 0} \frac{1}{t} \big[\mathcal{L}(u^t) - \mathcal{L}(u)\big] = \lim_{t \to 0} \frac{1}{t} \big[f(u^t) - f(u)\big]. \tag{8}$$

If f is differentiable, we have

$$f(u^t) - f(u) = f'(u + \theta(u^t - u))\big(u^t - u\big). \quad 0 < \theta < 1.$$

Therefore,

$$\lim_{t \to 0} \frac{1}{t} \big[f(u^t) - f(u)\big] = f'(u)v. \tag{9}$$

The computation of the left hand side of (8) depends on the structure of the differential operator $\mathcal{L}$. If $\mathcal{L}(u) = \Delta u$ we find

$$\lim_{t\to 0} \frac{1}{t}\big[\Delta u^t - \Delta u\big] = \Delta v.$$

Consider now the p-Laplacian $\mathcal{L}(u) = \mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big)$. We have

$$|\nabla u^t|^{p-2}\nabla u^t - |\nabla u|^{p-2}\nabla u = |\nabla u|^{p-2}(\nabla u^t - \nabla u) + \big(|\nabla u^t|^{p-2} - |\nabla u|^{p-2}\big)\nabla u^t.$$

Therefore,

$$\lim_{t\to 0} \frac{1}{t}\Big(|\nabla u^t|^{p-2}\nabla u^t - |\nabla u|^{p-2}\nabla u\Big) = |\nabla u|^{p-2}\nabla v + (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla v)\nabla u.$$

Hence, in this case, the equation corresponding to (8) for v reads as

$$\mathrm{div}\Big(|\nabla u|^{p-2}\nabla v + (p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla v)\nabla u\Big) = f'(u)v. \tag{10}$$

Now we recall the definition of generalized Monge-Ampére operators. Let $1 \le \kappa \le N$, and let $S_\kappa(u)$ be the κ-th elementary symmetric function of the eigenvalues of the Hessian matrix $H = D^2u = [u_{ij}]$ (that is, the sum of all principal minors of order κ of H). Clearly, we have $S_1(u) = \Delta u$ (Laplace operator) and $S_N(u) = \det[D^2u]$ (Monge-Ampére operator). Given a positive smooth function $f(t)$, we consider the problem

$$S_\kappa(u) = f(u) \;\text{ in }\; D, \quad u = 0 \;\text{ on }\; \partial D. \tag{11}$$

Suppose the domain $D \subset \mathbb{R}^N$ is bounded and smooth. In addition, for κ fixed such that $2 \le \kappa \le N$, we assume the following property:

$$\sigma_{\kappa-1} \ge \beta \quad \text{on } \; \partial D, \tag{P_κ}$$

where β is a positive constant and $\sigma_{\kappa-1}$ is the $(\kappa-1)$-th elementary symmetric function of the principal curvatures of ∂D with respect to its inner normal, see [4, 18]. If we denote by $\tau_1, \tau_2, \cdots, \tau_{N-1}$ the principal curvatures of the surface ∂D we have:

$$\sigma_1 = \sum_{1\le i\le N-1} \tau_i, \qquad \sigma_2 = \sum_{1\le i_1<i_2\le N-1} \tau_{i_1}\tau_{i_2}, \qquad \sigma_{N-1} = \tau_1\tau_2\cdots\tau_{N-1}.$$

Note that condition (P_N) means that Ω is strictly convex. Moreover, if Ω enjoys property (P_κ) then also $D^t = (I+tV)(D)$, for $|t|$ small, enjoys the same property (possibly with a smaller constant β). Finally, $f(t)$ is a positive smooth function such that problem (11) has a unique admissible solution. As usual, a solution is admissible if the operator $S_\kappa(u)$ is positive definite. In this situation, the solution u is negative in D and $\nu = \frac{\nabla u}{|\nabla u|}$ on the boundary ∂D. We refer to [4, 18] for a careful discussion of this problem.

It is convenient to define the matrix

$$T^{ij}_{\kappa-1}(u) = \frac{\partial S_\kappa(u)}{\partial u_{ij}}, \quad i,j = 1,\cdots,N. \tag{12}$$

We put $T_0(u) = I$, the identity matrix. The matrix $T_\kappa(u)$ is known as the κ-th Newton tensor associated with H. We have [13]

$$T_\kappa(u) = S_\kappa(u)I - T_{\kappa-1}(u)H, \quad \kappa = 1,\cdots,N-1.$$

Since H is symmetric, also T_κ is symmetric. It has several nice properties. For example, we have

$$\left(T^{ij}_\kappa(u)\right)_i = 0, \quad j = 1,\cdots,N, \tag{13}$$

where $\left(T^{ij}_\kappa(u)\right)_i = \frac{\partial T^{ij}_\kappa(u)}{\partial x^i}$, and here and in what follows, we use the summation convention over repeated indices from 1 to N. To prove (13), we recall the definition of the generalized Kronecker symbol

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_q \\ j_1 & j_2 & \cdots & j_q \end{pmatrix}, \quad 2 \le q \le N,$$

where $i_1,\cdots,i_q$ are distinct integers between 1 and N, and also $j_1,\cdots,j_q$ are distinct integers between 1 and N. The value of the symbol is 1 (respectively -1) if $(j_1,\cdots,j_q)$ is and even (respectively an odd) permutation of $(i_1,\cdots,i_q)$, and is 0 in all other cases. If $1 \le \kappa \le N-1$ we have (see [12])

$$T^{ij}_\kappa(u) = \frac{1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_\kappa & i \\ j_1 & j_2 & \cdots & j_\kappa & j \end{pmatrix} u_{i_1j_1}u_{i_2j_2}\cdots u_{i_\kappa j_\kappa}. \tag{14}$$

We find

$$\left(T^{ij}_\kappa(u)\right)_i = \frac{1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_\kappa & i \\ j_1 & j_2 & \cdots & j_\kappa & j \end{pmatrix} (u_{i_1j_1}u_{i_2j_2}\cdots u_{i_\kappa j_\kappa})_i.$$

Simplifying we can write

$$\left(T^{ij}_\kappa(u)\right)_i = \frac{1}{(\kappa-1)!}\begin{pmatrix} i_1 & i_2 & \cdots & i_\kappa & i \\ j_1 & j_2 & \cdots & j_\kappa & j \end{pmatrix} u_{i_1j_1i}u_{i_2j_2}\cdots u_{i_\kappa j_\kappa}.$$

We note that $u_{i_1j_1i}$ is symmetric with respect to i_1i, while the Kronecker symbol is skew-symmetric with respect to those indices. Thus, the sum over i_1i vanish, and (13) follows.

The proof in above can be extended to prove that, if also v is a smooth function, we have

$$\begin{pmatrix} i_1 & i_2 & \cdots & i_\kappa & i \\ j_1 & j_2 & \cdots & j_\kappa & j \end{pmatrix} \left(v_{i_1j_1}u_{i_2j_2}\cdots u_{i_\kappa j_\kappa}\right)_i = 0, \quad j = 1,\cdots,N. \tag{15}$$

We refer to Proposition 2.1 of [12] for details.

Another very interesting property is the following (see [12, 13])

$$\frac{1}{\kappa} T^{ij}_{\kappa-1}(u) u_{ij} = S_\kappa(u), \quad \kappa = 1, \cdots, N. \tag{16}$$

We are now ready to find the equation for v defined as in (6) with $\mathcal{L}(u) = S_\kappa(u)$. Let u^t be the (admissible) solution to problem (11) corresponding to D^t. Using (16) and (13) we have

$$\begin{aligned} S_\kappa(u^t) - S_\kappa(u) &= \frac{1}{\kappa}\Big(T^{ij}_{\kappa-1}(u^t)u^t_{ij} - T^{ij}_{\kappa-1}(u)u_{ij}\Big) \\ &= \frac{1}{\kappa}\Big(T^{ij}_{\kappa-1}(u^t)u^t_{i} - T^{ij}_{\kappa-1}(u)u_{i}\Big)_j \\ &= \frac{1}{\kappa}\Big[\Big(T^{ij}_{\kappa-1}(u)(u^t_i - u_i)\Big)_j + \Big(\big(T^{ij}_{\kappa-1}(u^t) - T^{ij}_{\kappa-1}(u)\big)u^t_i\Big)_j\Big]. \end{aligned} \tag{17}$$

We have

$$\lim_{t\to 0} \frac{1}{t} T^{ij}_{\kappa-1}(u)(u^t_i - u_i) = T^{ij}_{\kappa-1}(u) v_i.$$

Using (13) again we find

$$\lim_{t\to 0} \frac{1}{t} \Big(T^{ij}_{\kappa-1}(u)(u^t_i - u_i)\Big)_j = T^{ij}_{\kappa-1}(u) v_{ij}. \tag{18}$$

Moreover, using (14), we have

$$\begin{aligned} &\Big(\big(T^{ij}_{\kappa-1}(u^t) - T^{ij}_{\kappa-1}(u)\big)u^t_i\Big)_j \\ &= \frac{1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \Big(\big(u^t_{i_1j_1}u^t_{i_2j_2}\cdots u^t_{i_{\kappa-1}j_{\kappa-1}} - u_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}}\big)u^t_i\Big)_j. \end{aligned}$$

Hence,

$$\begin{aligned} &\lim_{t\to 0}\frac{1}{t}\Big(\big(T^{ij}_{\kappa-1}(u^t) - T^{ij}_{\kappa-1}(u)\big)u^t_i\Big)_j \\ &= \frac{1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \Big(\big(v_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}} + \cdots + u_{i_1j_1}u_{i_2j_2}\cdots v_{i_{\kappa-1}j_{\kappa-1}}\big)u_i\Big)_j \\ &= \frac{\kappa-1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \Big(v_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}}u_i\Big)_j \\ &= \frac{\kappa-1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j \end{pmatrix} \Big(\big(v_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}}\big)_j u_i + v_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}}u_{ij}\Big). \end{aligned}$$

By using (15) and changing conveniently the indices we find

$$\lim_{t\to 0}\frac{1}{t}\Big(\big(T^{ij}_{\kappa-1}(u^t)-T^{ij}_{\kappa-1}(u)\big)u^t_i\Big)_j = \frac{\kappa-1}{\kappa!}\begin{pmatrix} i_1 & i_2 & \cdots & i_{\kappa-1} & i \\ j_1 & j_2 & \cdots & j_{\kappa-1} & j\end{pmatrix} u_{i_1j_1}u_{i_2j_2}\cdots u_{i_{\kappa-1}j_{\kappa-1}}v_{ij}$$

$$= (\kappa-1)T^{ij}_{\kappa-1}(u)v_{ij},$$

where (14) has been used once more. From (17), (18) and the latter result we find

$$\lim_{t\to 0}\frac{S_\kappa(u^t)-S_\kappa(u)}{t} = \frac{1}{\kappa}\Big(T^{ij}_{\kappa-1}(u)v_{ij} + (\kappa-1)T^{ij}_{\kappa-1}(u)v_{ij}\Big) = T^{ij}_{\kappa-1}(u)v_{ij}.$$

Hence, recalling (9), we find the equation for v:

$$T^{ij}_{\kappa-1}(u)v_{ij} = f'(u)v. \qquad (19)$$

3 Duality results

In this section, we extend Theorem 1.1 to p-Laplace equations and to generalized Monge-Ampére equations.

3.1 p-Laplace equations

Let $D\subset\mathbb{R}^N$ be a bounded smooth domain, and let $f:\mathbb{R}\to\mathbb{R}$ be a C^1 positive function such that the problem

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(u) \ \text{ in } \ D, \quad u=0 \ \text{ on } \ \partial D, \qquad (20)$$

has a unique (negative) solution $u\in C^1(\bar{D})\cap W^{1,p}(D)$. For example, one can take, for $\tau<0$, $f(\tau)=(-\tau)^\alpha$, $0\le\alpha<p$, see [6]. We have

Theorem 3.1 *Let u be the solution to problem (20). Then the following statements are equivalent.*
(i) There is a constant c such that

$$|\nabla u| = c \quad \text{on } \ \partial D. \qquad (21)$$

(ii) There is a constant d such that

$$\int_D\Big(f(u)-\frac{1}{p-1}uf'(u)\Big)v\,dx = d\int_{\partial D} v\,d\sigma \qquad (22)$$

for all solutions v to equation (10).

Proof. Multiplying (10) by $-u$, integrating over D and recalling that $u = 0$ on ∂D we find

$$
\begin{aligned}
&-\int_D f'(u) v u \, dx = -\int_D \operatorname{div}\Big(|\nabla u|^{p-2} \nabla v + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u \Big) u \, dx \\
&= \int_D \Big(|\nabla u|^{p-2} \nabla v + (p-2) |\nabla u|^{p-4} (\nabla u \cdot \nabla v) \nabla u \Big) \cdot \nabla u \, dx \\
&= (p-1) \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx \\
&= (p-1) \int_{\partial D} |\nabla u|^{p-2} \nabla u \cdot \nu \, v \, d\sigma - (p-1) \int_D \operatorname{div}\big(|\nabla u|^{p-2} \nabla u \big) v \, dx.
\end{aligned}
$$

Since $\nu = \frac{\nabla u}{|\nabla u|}$ on ∂D, using equation (20), from the latter result we find

$$
\int_D \Big(f(u) - \frac{1}{p-1} f'(u) u \Big) v \, dx = \int_{\partial D} |\nabla u|^{p-1} v \, d\sigma. \tag{23}
$$

If (i) holds, equation (23) yields (22) with $d = c^{p-1}$.

If (ii) holds, from equations (22) and (23) we find

$$
\int_{\partial D} |\nabla u|^{p-1} v \, d\sigma = d \int_{\partial D} v \, d\sigma. \tag{24}
$$

Using the boundary condition (7) we have

$$
v = -\frac{\partial u}{\partial \nu} V \cdot \nu = -|\nabla u| \, V \cdot \nu.
$$

Therefore, from (24) we find

$$
\int_{\partial D} \big(|\nabla u|^p - d |\nabla u| \big) \, V \cdot \nu \, d\sigma = 0. \tag{25}
$$

Since V is arbitrary, we must have

$$
|\nabla u| \big(|\nabla u|^{p-1} - d \big) = 0 \quad \text{on} \quad \partial D.
$$

By Hopf's Lemma $|\nabla u| > 0$, hence, $|\nabla u| = d^{\frac{1}{p-1}}$ on ∂D. The theorem is proved.

3.2 Generalized Monge-Ampére equations

Let κ be an integer such that $1 \leq \kappa \leq N$. Let $D \subset \mathbb{R}^N$ be a bounded smooth domain satisfying property (P_κ), and let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 positive function such that the problem

$$
S_\kappa(u) = f(u) \ \text{ in } \ D, \quad u = 0 \ \text{ on } \ \partial D, \tag{26}
$$

has a unique admissible solution $u \in C^3(D) \cap C^1(\bar{D})$. We have

Theorem 3.2 *Let u be the admissible solution to problem (26). The following statements are equivalent.*
(i) There is a constant c such that

$$T^{ij}_{\kappa-1}(u)u_iu_j = c^2 \quad on \quad \partial D. \tag{27}$$

(ii) There is a constant d such that

$$\int_D \big(\kappa f(u) - uf'(u)\big)v\,dx = d\int_{\partial D}\frac{v}{|\nabla u|}\,d\sigma \tag{28}$$

hold for all solutions v to equation (19).

Proof. Multiplying (19) by $-u$, integrating over D, using (13) and recalling that $u = 0$ on ∂D we find

$$\begin{aligned} -\int_D uf'(u)v\,dx &= -\int_D uT^{ij}_{\kappa-1}(u)v_{ij}\,dx \\ &= -\int_D u\big(T^{ij}_{\kappa-1}(u)v_i\big)_j\,dx = \int_D u_jT^{ij}_{\kappa-1}(u)v_i\,dx. \end{aligned} \tag{29}$$

Integrating by parts and using (13) again we find

$$\int_D u_jT^{ij}_{\kappa-1}(u)v_i\,dx = \int_{\partial D} u_jT^{ij}_{\kappa-1}(u)\,v\,\nu^i\,d\sigma - \int_D u_{ij}T^{ij}_{\kappa-1}(u)\,v\;dx.$$

Since $T^{ij}_{\kappa-1}(u)u_{ij} = \kappa f(u)$ in D and $\nu^i|\nabla u| = u_i$ on ∂D, from the latter equation we find

$$\int_D u_jT^{ij}_{\kappa-1}(u)v_i\,dx = \int_{\partial D} T^{ij}_{\kappa-1}(u)u_iu_j\frac{v}{|\nabla u|}\,d\sigma - \int_D \kappa f(u)v\;dx. \tag{30}$$

From (29) and (30) it follows that

$$\int_D \big(\kappa f(u) - uf'(u)\big)v\,dx = \int_{\partial D} T^{ij}_{\kappa-1}(u)u_iu_j\frac{v}{|\nabla u|}\,d\sigma. \tag{31}$$

If (i) holds, equation (31) yields (28) with $d = c^2$.

If (ii) holds, from equations (28) and (31) we find

$$\int_{\partial D}\big[T^{ij}_{\kappa-1}(u)u_iu_j - d\big]\frac{v}{|\nabla u|}\,d\sigma = 0. \tag{32}$$

Finally, using the boundary condition (7) we get

$$\int_{\partial D}\big[T^{ij}_{\kappa-1}(u)u_iu_j - d\big]V\cdot\nu\,d\sigma = 0.$$

Since V is arbitrary, (27) follows with $c^2 = d$. The theorem is proved.

Let us recall a result from [1].

Theorem 3.3 *Let D be a bounded convex domain in the plane and let c be a constant. If there exists a convex solution u to the Dirichlet problem*

$$S_2(u) = u_{11}u_{22} - u_{12}^2 = 1 \quad in\ D, \quad u = 0 \quad on\ \partial D \tag{33}$$

such that u satisfies the additional condition

$$u_{22}u_1^2 + u_{11}u_2^2 - 2u_{12}u_1u_2 = c^2 \quad on\ \partial D, \tag{34}$$

then D must be an ellipse.

Proof. See Theorem 2.4 of [1].

Corollary 3.4 *Let D be a bounded convex domain in the plane and let c be a constant. If there exists a convex solution u to problem (33) such that the integral equations*

$$2\int_D v\,dx = c^2 \int_{\partial D} \frac{v}{|\nabla u|}\,ds, \tag{35}$$

hold for all solutions v to the equation

$$u_{22}v_{11} + u_{11}v_{22} - 2u_{12}v_{12} = 0 \quad in\ \ D, \tag{36}$$

then, D is an ellipse.

Proof. Since $T_1^{11}(u) = u_{22}$, $T_1^{12}(u) = T_1^{21}(u) = -u_{12}$ and $T_1^{22}(u) = u_{11}$, equation (36) can be written as $T_1^{ij}(u)v_{ij} = 0$, and condition (34) can be written as $T_1^{ij}u_iu_j = c^2$. Hence, the corollary follows from Theorem 3.2 and Theorem 3.3.

4 Minimization of functionals

In this section we present a motivation of the overdetermined conditions (2) and (27).

4.1 p-Laplace equations

Let $D \subset \mathbb{R}^N$ be a bounded smooth domain, and recall the problem (20) below

$$\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = f(u) \;\text{ in }\; D, \;\; u = 0 \;\text{ on }\; \partial D, \tag{37}$$

where f is a positive function such that problem (37) has a unique (negative) solution. Given D and the corresponding solution u to problem (37), we consider the functional

$$J(D) = \int_D \Big(|\nabla u|^p + p\int_0^u f(\tau)\,d\tau\Big)dx. \tag{38}$$

Theorem 4.1 *Let $J(D)$ be defined as in (38). If $\hat{D}$ is a minimum of $J(D)$ among all domains D having the same measure as $\hat{D}$, then $|\nabla u|$ is constant on $\partial\hat{D}$.*

Proof. To prove the theorem, we use the notion of domain derivative. Recall that I is the identity map. Let $V : \mathbb{R}^N \to \mathbb{R}^N$ be a smooth vector field, and let $D^t = (I + tV)\hat{D}$ be a deformation of $\hat{D}$. By the well known Lagrange principle, $\hat{D}$ is a stationary point of the functional

$$I(D) \equiv J(D) + \lambda(K(D) - \mu), \quad K(D) = |D|, \quad \mu = |\hat{D}|,$$

where λ is a real parameter. Since $\hat{D}$ is a stationary point of $I(D)$, we must have $dI(\hat{D}, V) = 0$ for every vector field V. Clearly,

$$dI(\hat{D}, V) \equiv dJ(\hat{D}, V) + \lambda\, dK(\hat{D}, V). \tag{39}$$

We compute first $dJ(\hat{D}, V)$. Let u be the solution of problem (37) with $D = \hat{D}$, and let u^t be the solution of problem (37) corresponding to D^t. We have

$$\begin{aligned}
dJ(\hat{D}, V) &= \lim_{t\to 0} \frac{J(D^t) - J(\hat{D})}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\Big[\int_{D^t} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx - \int_{\hat{D}} \Big(|\nabla u|^p + p\int_0^{u} f(\tau)\, d\tau\Big) dx\Big] \\
&= \lim_{t\to 0} \frac{1}{t}\Big[\int_{D^t} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx - \int_{\hat{D}} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx\Big] \\
&+ \lim_{t\to 0} \frac{1}{t}\Big[\int_{\hat{D}} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx - \int_{\hat{D}} \Big(|\nabla u|^p + p\int_0^{u} f(\tau)\, d\tau\Big) dx\Big].
\end{aligned}$$

Since $u = 0$ on $\partial\hat{D}$ we find

$$\begin{aligned}
&\lim_{t\to 0} \frac{1}{t}\Big[\int_{D^t} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx - \int_{\hat{D}} \Big(|\nabla u^t|^p + p\int_0^{u^t} f(\tau)\, d\tau\Big) dx\Big] \\
&= \int_{\partial\hat{D}} \Big(|\nabla u|^p + p\int_0^{u} f(\tau)\, d\tau\Big)\, V\cdot\nu\, d\sigma = \int_{\partial\hat{D}} |\nabla u|^p\, V\cdot\nu\, d\sigma.
\end{aligned}$$

Therefore, we find

$$\begin{aligned}
dJ(\hat{D}, V) &= \int_{\partial\hat{D}} |\nabla u|^p\, V\cdot\nu\, d\sigma \\
&+ \lim_{t\to 0}\int_{\hat{D}} \frac{|\nabla u^t|^p - |\nabla u|^p}{t} dx + p\lim_{t\to 0}\int_{\hat{D}} \frac{\int_u^{u^t} f(\tau)\, d\tau}{t} dx \\
&= \int_{\partial\hat{D}} |\nabla u|^p\, V\cdot\nu\, d\sigma + p\Big(\int_{\hat{D}} |\nabla u|^{p-2}\nabla u\cdot\nabla v\, dx + \int_{\hat{D}} f(u) v\, dx\Big),
\end{aligned} \tag{40}$$

where v is defined as

$$v(x) = \lim_{t\to 0} \frac{u^t(x) - u(x)}{t}.$$

Integrating the equation

$$-v\,\mathrm{div}\big(|\nabla u|^{p-2}\nabla u\big) = -f(u)v$$

over $\hat{D}$ we find

$$-\int_{\partial\hat{D}} v|\nabla u|^{p-2}\nabla u\cdot\nu\,d\sigma + \int_{\hat{D}}|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx = -\int_{\hat{D}} f(u)v\,dx.$$

Recalling that $\nabla u = |\nabla u|\,\nu$ on ∂D and using the boundary condition (7), from the latter equation we find

$$\int_{\hat{D}}|\nabla u|^{p-2}\nabla u\cdot\nabla v\,dx + \int_{\hat{D}} f(u)v\,dx = -\int_{\partial\hat{D}}|\nabla u|^p\,V\cdot\nu\,d\sigma.$$

By (40) and the latter equation we find
(41)

$$dJ(\hat{D},V) = \int_{\partial\hat{D}}|\nabla u|^p\,V\cdot\nu\,d\sigma - p\int_{\partial\hat{D}}|\nabla u|^p\,V\cdot\nu\,d\sigma = -(p-1)\int_{\partial\hat{D}}|\nabla u|^p\,V\cdot\nu\,d\sigma.$$

On the other hand (see [16] page 652 formula (12) with $C(u)=1$), we have

$$dK(\hat{D},V) = \int_{\partial\hat{D}} V\cdot\nu\,d\sigma. \tag{42}$$

Insertion of (41) and (42) into (39) yields

$$\begin{aligned} dI(\hat{D},V) &= -(p-1)\int_{\partial\hat{D}}|\nabla u|^p\,V\cdot\nu\,d\sigma + \lambda\int_{\partial\hat{D}} V\cdot\nu\,d\sigma \\ &= \int_{\partial\hat{D}}\Big(-(p-1)|\nabla u|^p + \lambda\Big)\,V\cdot\nu\,d\sigma. \end{aligned}$$

Since $dI(\hat{D},V)=0$ for every vector field V, it follows that $|\nabla u|^p = \frac{\lambda}{p-1}$. Therefore, $|\nabla u|$ is a constant on $\partial\hat{D}$, and the theorem is proved.

4.2 Generalized Monge-Ampére equations

Now we prove a similar result for generalized Monge-Ampére equations. Assume the domain D bounded, smooth and having the property (P_κ). Let u be an admissible solution to the problem

$$S_\kappa(u) = f(u) \quad \text{in } D, \quad u=0 \text{ on } \partial D. \tag{43}$$

Here $1\le\kappa\le N$ and $f(t)>0$. Consider the functional

$$E(D) = \int_D\Big(T^{ij}_{\kappa-1}(u)u_iu_j + (\kappa+\kappa^2)\int_0^u f(\tau)\,d\tau\Big)dx, \tag{44}$$

where u is an admissible solution to problem (43).

Theorem 4.2 *Let $E(D)$ be defined as in (44). If $\hat{D}$ is a minimum of $E(D)$ among all domains D having the property (P_κ) and having the same measure as $\hat{D}$, then we have*

$$T^{ij}_{\kappa-1}(u)u_iu_j = \text{ constant } \quad \text{on } \partial\hat{D}.$$

Proof. Let us find a different formulation for $E(D)$. If we multiply (43) by u and use (16) we have

$$T^{ij}_{\kappa-1}(u)u_{ij}u = \kappa f(u)u.$$

Integration over D yields

$$-\int_D T^{ij}_{\kappa-1}(u)u_iu_j\,dx = \kappa\int_D f(u)u\,dx.$$

Hence, the functional defined by (44) can be rewritten as

$$E(D) = \int_D \Big(-\kappa f(u)u + (\kappa+\kappa^2)\int_0^u f(\tau)\,d\tau\Big)dx.$$

From now on, we shall use this formula for $E(D)$.

By the well known Lagrange principle, $\hat{D}$ is a stationary point of the functional

$$I(D) \equiv E(D) + \lambda(K(D)-\mu), \quad K(D) = |D|, \quad \mu = |\hat{D}|,$$

where λ is a real parameter. For a smooth vector field V, let $D^t = (I+tV)\hat{D}$ be a deformation of $\hat{D}$. We must have $dI(\hat{D},V) = 0$ for every vector field V. Clearly,

$$dI(\hat{D},V) \equiv dE(\hat{D},V) + \lambda\,dK(\hat{D},V). \tag{45}$$

If u^t is the solution to problem (43) corresponding to D^t, we compute

$$\begin{aligned} dE(\hat{D},V) &= \int_{\partial\hat{D}} \Big(-\kappa f(u)u + (\kappa+\kappa^2)\int_0^u f(\tau)\,d\tau\Big)V\cdot\nu\,d\sigma \\ &+ \lim_{t\to 0}\frac{1}{t}\int_{\hat{D}}\Big(-\kappa\big(f(u^t)u^t - f(u)u\big) + (\kappa+\kappa^2)\int_u^{u^t} f(\tau)\,d\tau\Big)\,dx. \end{aligned}$$

Since $u=0$ on $\partial\hat{D}$, the first integral vanishes. Hence,

$$\begin{aligned} dE(\hat{D},V) &= \int_{\hat{D}}\Big(-\kappa\big(f'(u)vu + f(u)v\big) + (\kappa+\kappa^2)f(u)v\Big)\,dx \\ &= \int_{\hat{D}}\big(-\kappa f'(u)vu + \kappa^2 f(u)v\big)\,dx. \end{aligned} \tag{46}$$

As usual, the function v is defined as

$$v(x) = \lim_{t\to 0}\frac{u^t(x)-u(x)}{t}.$$

Now we multiply equation (19) by $-u$ and integrate over $\hat{D}$. We find

$$-\int_{\hat{D}} f'(u)vu\,dx = -\int_{\hat{D}} T^{ij}_{\kappa-1}(u)v_{ij}u\,dx = \int_{\hat{D}} T^{ij}_{\kappa-1}(u)u_i v_j\,dx$$
$$= \int_{\partial\hat{D}} T^{ij}_{\kappa-1}(u)u_i\nu^j v\,d\sigma - \int_{\hat{D}} T^{ij}_{\kappa-1}(u)u_{ij}v\,dx.$$

Since $\nu^j|\nabla u| = u_j$ and $v = -|\nabla u|\,V\cdot\nu$ on $\partial\hat{D}$, and $T^{ij}_{\kappa-1}(u)u_{ij} = \kappa f(u)$ in $\hat{D}$, from the latter equation we find

$$-\int_{\hat{D}} \kappa f'(u)vu\,dx = -\kappa\int_{\partial\hat{D}} T^{ij}_{\kappa-1}(u)u_i u_j\,V\cdot\nu\,d\sigma - \int_{\hat{D}} \kappa^2 f(u)v\,dx.$$

In view of the latter result, from (46) we get

$$dE(\hat{D},V) = -\kappa\int_{\partial\hat{D}} T^{ij}_{\kappa-1}(u)u_i u_j\,V\cdot\nu\,d\sigma. \tag{47}$$

Insertion of (47) and (42) into (45) yields

$$dI(\hat{D},V) = \int_{\partial\hat{D}} \left(-\kappa T^{ij}_{\kappa-1}(u)u_i u_j + \lambda\right) V\cdot\nu\,d\sigma.$$

Since $dI(\hat{D},V) = 0$ for every vector field V, it follows that $T^{ij}_{\kappa-1}(u)u_i u_j = \frac{\lambda}{\kappa}$. Therefore, $T^{ij}_{\kappa-1}(u)u_i u_j$ is a constant on $\partial\hat{D}$, and the theorem is proved.

Corollary 4.3 *Let D be a convex planar domain, let u be a convex solution to problem (43) with $N = \kappa = 2$ and $f = 1$. If $E(D)$ is the corresponding functional defined as in (44), and if $\hat{D}$ is a minimum of $E(D)$ among all convex domains D having the same measure as $\hat{D}$, then D is an ellipse.*

Proof. It follows from Theorems 4.2 and 3.3.

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