Existence of Continuous Eigenvalues for a Class of Parametric Problems Involving the $(p, 2)$-Laplacian Operator*

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Abstract

We discuss a parametric eigenvalue problem, where the differential operator is of $(p, 2)$-Laplacian type. We show that, when $p \neq 2$, the spectrum of the operator is a half line, with the end point formulated in terms of the parameter and the principal eigenvalue of the Laplacian with zero Dirichlet boundary conditions. Two cases are considered corresponding to $p > 2$ and $p < 2$, and the methods that are applied are variational. In the former case, the direct method is applied, whereas in the latter case, the fibering method of Pohozaev is used. We will also discuss *a priori* bounds and regularity of the eigenfunctions. In particular, we will show that, when the eigenvalue tends towards the end point of the half line, the supremum norm of the corresponding eigenfunction tends to zero in the case of $p > 2$, and to infinity in the case of $p < 2$.

Keywords: fibering method, continuous eigenvalues, p-Laplacian.

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1 Introduction

The reaction-diffusion equation:

$$\frac{\partial u}{\partial t} - \nabla \cdot (\mathcal{H}(u)\nabla u) = f(x, u), \tag{1}$$

in which $\mathcal{H}(u) = a|\nabla u|^{p-2} + b|\nabla u|^{q-2}$ for positive constants a and b, has been used to model physical phenomena that arise in biophysics [11], plasma physics [27], and chemical reaction design [2]. Typically, in these models:

- $u(x, t)$ stands for concentration;
- $\mathcal{H}(u)\nabla u$ stands for the diffusion, with $\mathcal{H}(u)$ denoting the coefficient of diffusion;
- $f(x, u)$ denotes the reaction term related to the source and loss processes.

The steady-state version of (1) becomes the following quasilinear partial differential equation:

$$-a\Delta_p u - b\Delta_q u = f(x, u), \tag{2}$$

where Δ_s denotes the s-Laplacian operator $\Delta_s u := \nabla \cdot (|\nabla u|^{s-2}\nabla u)$ for $s \in (1, \infty)$.

Differential equations of type (2) have attracted a great deal of attention in recent years, see, e. g., [4, 8, 31, 32, 30, 5, 6, 13, 18, 3] for scalar equations, and [17, 7] for systems. When $p = q$, equation (2) becomes $-\Delta_p u = g(x, u)$, which has been studied extensively in the literature. As such, the more curious case is when $p \neq q$.

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Figueiredo [12] has studied a differential equation of the following type:

$$-\nabla\cdot\left(\mathcal{A}(|\nabla u|^q)\,|\nabla u|^{q-2}\nabla u\right)=h(x,u),\quad \text{in } \mathbb{R}^n, \tag{3}$$

where the function $\mathcal{A}:[0,\infty)\to[0,\infty)$ satisfies certain growth conditions, and shown that the equation has positive solutions. A particular choice of $\mathcal{A}$ which conforms to the required constraints is:

$$\mathcal{A}(\xi)=b+a\,\xi^{\frac{p-q}{q}},\quad q<p,\quad a,b\in\mathbb{R}^+. \tag{4}$$

Substituting $\mathcal{A}(\xi)$ in (3) yields (2). In particular, when $a=b=1$, we get:

$$-\Delta_p u-\Delta u=h(x,u). \tag{5}$$

The differential operator $-\Delta_p-\Delta$ in (5) is called the $(p,2)$-Laplacian. Recently, in a series of papers, equation (5) has been investigated for $p>2$, under the boundary condition $u=0$, and the assumption that $\Omega\subset\mathbb{R}^n$ is a bounded C^2 domain. In [23], the authors impose certain conditions on the reaction term $h(x,u)$ to make equation (5) resonant at $\pm\infty$ and zero. Using variational methods and critical groups, they obtain existence and multiplicity results. In [14], the authors consider the case with a reaction term $h(x,u)$ which is superlinear in the positive direction (without satisfying the Ambrosetti-Rabinowitz condition) and sublinear resonant in the negative direction. They apply Morse theory and variational methods to establish existence of at least three non-trivial smooth solutions. In [22], the authors consider (5) when the reaction term takes the following form (see also [21]):

$$-\Delta_p u-\Delta u=\lambda|u|^{p-2}u+f(x,u),\quad \text{in } \Omega.$$

Here $\lambda>0$ is a parameter, and the function $f(x,u)$ is a Carathéodory perturbation. They obtain existence and multiplicity results in the case of λ being near the principal eigenvalue $\hat{\lambda}_1$ of the p-Laplacian operator relative to $W_0^{1,p}(\Omega)$. Finally, we mention [1], in which the authors consider (4) with $a=1, b=\mu\geq 0$, and $q=2$. In this case, equation (3) becomes $-\Delta_p u-\mu\Delta u=h(x,u)$. They use variational methods and Morse theory to prove two multiplicity results providing precise sign information for all the solutions (both constant sign and nodal solutions).

Using critical point theory, truncation and comparison techniques, and Morse theory, Papageorgiou and Rădulescu [20] proved multiplicity results for (5) for both $p>2$ and $p<2$. In the latter case, they also used the Lyapunoff-Schmidt reduction method, together with certain conditions on the reaction term $h(x,u)$,

In the current paper, we are interested in the case when $p\in(1,\infty)\setminus\{2\}$, $a=t, b=1-t, q=2$, in (4), and $h(x,u)=\lambda u$ in (3), where $t\in[0,1]$, and λ is unknown. In this setting, the differential operator turns into a *convex* combination of $-\Delta_p$ and $-\Delta$, and induces a class of parametric eigenvalue problems:

$$\begin{cases} -t\Delta_p u-(1-t)\Delta u=\lambda u, & \text{in } D,\\ u=0, & \text{on } \partial D, \end{cases} \tag{6}$$

in which, $D\subseteq\mathbb{R}^n$ is a smooth bounded domain.

We will show that the set of eigenvalues of (6) is continuous for $t\in(0,1]$. In fact, if λ_1 is the first eigenvalue of $-\Delta$, then we will prove that the spectrum of (6) is the interval $((1-t)\lambda_1,\infty)$, even when t is very close to zero. This result is quite intriguing because when t approaches zero, the differential operator:

$$\mathfrak{C}_t:=-t\Delta_p-(1-t)\Delta$$

approaches $-\Delta$. Recall that the spectrum of the Laplacian is a discrete set:

$$\sigma(-\Delta)=\{\lambda_j\mid j\in\mathbb{N}\},\ \text{where } \lambda_1<\lambda_2\leq\lambda_3\leq\lambda_4\leq\cdots\to\infty.$$

In other words, when the convex parameter t moves from 1 to 0 in the interval $[0,1]$, the spectrum $\sigma(\mathfrak{C}_t)$ will keep containing the interval $[\lambda_1,\infty)$ until t takes the exact value 0, at which point $\sigma(\mathfrak{C}_t)$ collapses into the discrete set $\sigma(-\Delta)$.

In two recent papers [19, 10], the following eigenvalue problem has been investigated:

$$\begin{cases} -\Delta_p u - \Delta u = \lambda u, & \text{in } D, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial D, \end{cases}$$

where ν denotes the unit outward normal to the boundary ∂D. In [19], the author considers the case $p > 2$, and proves that the spectrum is $\{0\} \cup (\lambda_1, \infty)$, where:

$$\lambda_1 = \inf_{\left\{u \in W^{1,p}(D)\setminus\{0\},\ \int_D u\, dx = 0\right\}} \frac{\int_D |\nabla u|^2 dx}{\int_D u^2 dx}.$$

On the other hand, in [10], the authors prove that when $p \in (1,2)$, the spectrum is $\{0\} \cup (\lambda_1^N, \infty)$, where λ_1^N denotes the first non-zero eigenvalue of $-\Delta$ with respect to the Neumann boundary condition. Note that when $p > 2$, $W^{1,p}(D) \subseteq W^{1,2}(D)$, hence $\lambda_1^N \leq \lambda_1$.

Our approach toward solving the eigenvalue problem (6) differs from the ones taken in [19] and [10], in that ours is based on the ***fibering method*** that was introduced in the early 1990s by S. Pohozaev [24], whereas the approach in [19] is based on the direct method of calculus of variations, and in [10], the Nehari-manifold. The fibering method is more powerful than that of the Nehari-manifold, as it is applicable to a much broader range of boundary value problems than we discuss here (see, e. g., [28, 29]).

The following is a summary of what is known about the spectrum of some of the related operators:

(i) $\sigma(-\Delta) = \{\lambda_j \mid j \in \mathbb{N}\}$, in which $\lambda_1 < \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots \to \infty$, with respect to both Dirichlet and Neumann boundary conditions. In the latter case, $\lambda_1 = 0$ and $\lambda_2 < \lambda_3$.

(ii) $\sigma(-\Delta_p - \Delta) = \begin{cases} \{0\} \cup (\lambda_1, \infty), & 2 < p, \\ \{0\} \cup (\lambda_1^N, \infty), & 1 < p < 2, \end{cases}$ with respect to the Neumann boundary conditions.

(iii) $\sigma(-\Delta_p) \subseteq [0, \infty)$, provided that $p \in (\frac{2n}{n+2}, \infty) \setminus \{2\}$, with respect to the Neumann boundary conditions. In this case, the zero eigenvalue is isolated.

1.1 Our main results

The current paper has three main theorems. We will present the proofs in Sections 2 and 3. The first of these theorems is:

Theorem 1.1. *Let $p \in (1, \infty) \setminus \{2\}$, $t \in (0, 1)$, and assume that λ_1 denotes the first eigenvalue of $-\Delta$ with respect to the Dirichlet boundary condition on ∂D. Then $\sigma(\mathfrak{C}_t) = ((1 - t)\lambda_1, \infty)$.*

Figure 1 on the following page depicts the claim of Theorem 1.1. We prove the theorem using variational methods. For this purpose, we will consider an energy functional associated with (6), and prove that the critical points of this functional will give rise to non-trivial solutions of (6). For $p > 2$, the energy functional is coercive, hence the direct method applies. The main challenge lies in the case $p < 2$, where the lack of coercivity renders the direct method ineffective. Hence, we shall apply the fibering method.

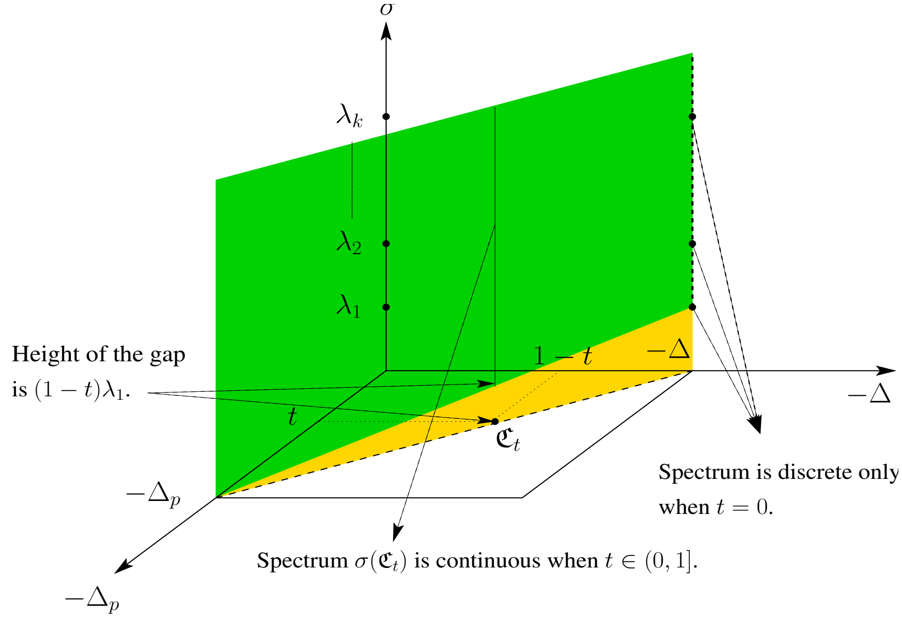
In Section 3, we will derive *a priori* bounds and regularity results on the eigenfunctions. We will show that the behavior of the eigenfunctions are totally different between the cases of $p > 2$ and $p \in (1, 2)$:

Theorem 1.2. *Assume that $p \in (2, \infty)$, and let $u \in W_0^{1,p}(D)$ be a non-trivial solution of (6). Then the following hold:*

(i) $u \in C^{2,\alpha}_{loc} \cap L^\infty(D)$, and $u \in C^\infty(D \setminus \{\nabla u = 0\})$.

(ii) $\|u\|_{W_0^{1,p}(D)} \leq \left(\frac{\lambda - (1-t)\lambda_1}{\lambda_1 t}\right)^{1/(p-2)} |D|^{1/p}$.

Figure 1 Dynamics of the spectrum as t ranges over $[0, 1]$.



(iii) $\sup_D |u| \le \frac{C}{\lambda_1^{1/2}} \left(\frac{\lambda-(1-t)\lambda_1}{\lambda_1 t}\right)^{1/(p-2)} |D|^{1/2}$. *In particular,* $\sup_D |u| \to 0$ *as* $\lambda \downarrow (1-t)\lambda_1$.[1]

Theorem 1.3. *Assume that* $p \in (1, 2)$ *and let* $u \in W_0^{1,2}(D)$ *solve (6). Then the following hold:*

(i) $u \in C^{1,\alpha}_{loc} \cap L^\infty(D)$, *and* $u \in C^\infty(D \setminus \{\nabla u = 0\})$.

(ii) For any non-trivial solution u, there is a constant $C = C(\lambda, p, n, D)$ *such that:*

$$\|u\|_{W_0^{1,p}(D)} \ge \left(\int_D |u|^p \, dx\right)^{1/p} \ge C\left(\frac{t\lambda_p}{\lambda - \lambda_1(1-t)}\right)^{1/(2-p)}.$$

Moreover, $C > 0$ *if* $\lambda > 0$.

(iii) $\sup_D |u| \to \infty$ *as* $\lambda \downarrow (1-t)\lambda_1$.

2 Proof of Theorem 1.1.

Definition 2.1. Eigenpairs of (6) are defined as follows:

$\boldsymbol{p > 2}$ **:** We say that $(\lambda, u) \in \mathbb{R} \times (W_0^{1,p}(D) \setminus \{0\})$ is an eigenpair of (6) provided that the following integral equation holds:

$$t\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + (1-t)\int_D \nabla u \cdot \nabla v \, dx = \lambda \int_D uv \, dx, \quad \forall v \in W_0^{1,p}(D).$$

$\boldsymbol{p \in (1,2)}$ **:** We say that $(\lambda, u) \in \mathbb{R} \times (W_0^{1,2}(D) \setminus \{0\})$ is an eigenpair of (6) provided that the following integral equation holds:

$$t\int_D |\nabla u|^{p-2}\nabla u \cdot \nabla v \, dx + (1-t)\int_D \nabla u \cdot \nabla v \, dx = \lambda \int_D uv \, dx, \quad \forall v \in W_0^{1,2}(D).$$

[1] Here, C is as in Lemma 3.1 on page 10.

Now, suppose that (λ, u) is an eigenpair of (6), and $t \in (0,1)$. From (6), we obtain:

$$t \int_D |\nabla u|^p \, dx + (1-t) \int_D |\nabla u|^2 \, dx = \lambda \int_D u^2 \, dx,$$

which implies that $(1-t) \int_D |\nabla u|^2 \, dx < \lambda \int_D u^2 \, dx$. On the other hand, for the first eigenvalue λ_1 of $-\Delta$ with Dirichlet boundary conditions on ∂D, we have $\lambda_1 \int_D u^2 \, dx \leq \int_D |\nabla u|^2 \, dx$.[2] So, we deduce that $(1-t)\lambda_1 < \lambda$.

In other words, if $\lambda \in [0, (1-t)\lambda_1]$ then $\lambda \notin \sigma(\mathfrak{C}_t)$, in which $\sigma(\mathfrak{C}_t)$ denotes the spectrum of $-t\Delta_p - (1-t)\Delta$ with Dirichlet boundary conditions on ∂D. In what follows, we will prove that every $\lambda > (1-t)\lambda_1$ is indeed an eigenvalue.

2.1 Proof of Theorem 1.1 for $p > 2$

The proof for $p > 2$ is straightforward, see also [25]. Henceforth, we fix $t \in (0,1)$ and $\lambda > (1-t)\lambda_1$. Note that there exists a function $\psi \in W_0^{1,2}(D)$ for which:

$$\lambda_1 < \frac{\int_D |\nabla \psi|^2 \, dx}{\int_D \psi^2 \, dx} < \frac{\lambda}{1-t}.$$

Since $C_0^\infty(D)$ is dense in $W_0^{1,2}(D)$, we can find $\phi \in C_0^\infty(D)$ such that:

$$\lambda_1 < \frac{\int_D |\nabla \phi|^2 \, dx}{\int_D \phi^2 \, dx} < \frac{\lambda}{1-t}. \tag{7}$$

Note that $\phi \in W_0^{1,p}(D)$.

Remember that our task is to prove that λ is an eigenvalue of (6). Consider the functional $\Phi : W_0^{1,p}(D) \to \mathbb{R}$:

$$\Phi(u) := \frac{t}{p} \int_D |\nabla u|^p \, dx + \frac{1-t}{2} \int_D |\nabla u|^2 \, dx - \frac{\lambda}{2} \int_D u^2 \, dx.$$

Since $p > 2$, the functional Φ is coercive. Now, consider the minimization problem:

$$\inf_{u \in W_0^{1,p}(D)} \Phi(u) =: m. \tag{8}$$

Let (u_n) be a minimizing sequence for (8). Since Φ is coercive, there exists a function $u \in W_0^{1,p}(D)$, and a subsequence of (u_n)—still denoted (u_n)—such that $u_n \rightharpoonup u$ in $W_0^{1,p}(D)$ and in $W_0^{1,2}(D)$. From the compact embedding $W_0^{1,2}(D) \to L^2(D)$, it follows that $u_n \to u$ in $L^2(D)$. Observe that Φ is weakly lower semi-continuous. So, $\Phi(u) \leq \liminf_{n\to\infty} \Phi(u_n) = m$. This implies $\Phi(u) = m$. For u to qualify as an eigenfunction corresponding to λ, it has to be non-trivial, i. e., $u \neq 0$. In that case, (λ, u) will be an eigenpair of (6), because $\Phi \in C^1(W_0^{1,p}(D), \mathbb{R})$.

To seek a contradiction, we assume that $m = 0$, which implies that $\Phi(v) \geq 0$ for every $v \in W_0^{1,p}(D)$. So, it suffices to find a $v \in W_0^{1,p}(D)$ such that $\Phi(v) < 0$. Let $\xi > 0$, and consider $\xi\phi$, where $\phi \in W_0^{1,p}(D)$ is the function satisfying (7). We have:

$$\begin{aligned} \Phi(\xi\phi) &= \frac{t\xi^p}{p} \int_D |\nabla \phi|^p \, dx + \frac{(1-t)\xi^2}{2} \int_D |\nabla \phi|^2 \, dx - \frac{\lambda \xi^2}{2} \int_D \phi^2 \, dx \\ &\leq \frac{t\xi^p}{p} \int_D |\nabla \phi|^p \, dx + \frac{\xi^2}{2} N, \end{aligned} \tag{9}$$

where, according to (7):

$$N := (1-t) \int_D |\nabla \phi|^2 \, dx - \lambda \int_D \phi^2 \, dx < 0.$$

[2] Note that if $p > 2$, then $W_0^{1,p}(D) \subseteq W_0^{1,2}(D)$. Hence, regardless of which value p takes, we would have $u \in W_0^{1,2}(D)$.

We choose ξ such that

$$\xi < \left(\frac{-pN}{2t \int_D |\nabla \phi|^p \, dx} \right)^{\frac{1}{p-2}} .$$

This choice of ξ, in conjunction with (9), guarantees $\Phi(\xi\phi) < 0$, which is in contradiction with $m = 0$. This completes the proof of the theorem for $p > 2$. □

2.2 Proof of Theorem 1.1 for $1 < p < 2$

Since $p < 2$, the appropriate function space would be $W_0^{1,2}(D)$. Note that in this case $W_0^{1,2}(D) \subseteq W_0^{1,p}(D)$. Henceforth, $\| \cdot \|_{1,p}$ denotes the norm in $W_0^{1,p}(D)$, i. e.:

$$\|u\|_{1,p} := \left(\int_D |\nabla u|^p \, dx \right)^{\frac{1}{p}} .$$

The notation $\|u\|_p$, as usual, is used for denoting the $L^p(D)$-norm. Let us introduce

$$H(u) := \lambda \int_D u^2 \, dx - (1-t) \int_D |\nabla u|^2 \, dx,$$

and observe that H is a 2-homogeneous functional, i. e., $H(\xi u) = \xi^2 H(u)$. We can write:

$$\Phi(u) = \frac{t}{p} \|u\|_{1,p}^p - \frac{1}{2} H(u).$$

Unlike the case of $p > 2$, here, Φ is not coercive. Hence, the direct method is not applicable. Instead, we use the fibering method as presented in the following theorem. We include the proof for the relevant insights it offers.

Theorem 2.1. *Let X be a real Banach space. Let $M : \mathcal{U} \subseteq X \to \mathbb{R}$ be differentiable and satisfy the non-degeneracy condition:*

$$\exists c \in \mathbb{R}, \forall u \in M^{-1}(c) : \langle M'(u), u \rangle \neq 0, \tag{10}$$

in which $\mathcal{U}$ is an open set and $M^{-1}(c) := \{u \in \mathcal{U} \mid M(u) = c\}$. Let $\Phi \in C^1(X \setminus \{0\})$, and define $\Psi : \mathbb{R} \times X \to \mathbb{R}$ by $\Psi(r, u) = \Phi(ru)$. Suppose $(\hat{r}, \hat{u}) \in \mathbb{R} \times M^{-1}(c)$, with $\hat{r}\hat{u} \neq 0$, is a (constrained) critical point of Ψ relative to $\mathbb{R} \times M^{-1}(c)$. Then, $\hat{v} = \hat{r}\hat{u}$ is a non-zero critical point of Φ, i. e. $\Phi'(\hat{v}) = 0$ in X^*.

Proof. Since $(\hat{r}, \hat{u})$ is a critical point of Ψ relative to $(\mathbb{R} \setminus \{0\}) \times M^{-1}(c)$, we can apply the Lagrange multiplier rule that ensures existence of $\mu \in \mathbb{R}$ such that:

$$\frac{\partial \Psi}{\partial r}(\hat{r}, \hat{u}) = 0, \tag{11}$$

and

$$\frac{\partial \Psi}{\partial u}(\hat{r}, \hat{u}) = \mu M'(\hat{u}), \quad \text{in } X^*. \tag{12}$$

Equations (11) and (12) are called *the fibering equations*. From (12), we obtain:

$$\langle \frac{\partial \Psi}{\partial u}(\hat{r}, \hat{u}), \hat{u} \rangle = \mu \langle M'(\hat{u}), \hat{u} \rangle.$$

On the other hand, we have:

$$\langle \frac{\partial \Psi}{\partial u}(\hat{r}, \hat{u}), \hat{u} \rangle = \hat{r} \langle \Phi'(\hat{r}\hat{u}), \hat{u} \rangle = \hat{r} \frac{\partial \Psi}{\partial r}(\hat{r}, \hat{u}).$$

Thus, we get:

$$\hat{r} \frac{\partial \Psi}{\partial r}(\hat{r}, \hat{u}) = \mu \langle M'(\hat{u}), \hat{u} \rangle. \tag{13}$$

From (11) and (13) we deduce that $\mu \langle M'(\hat{u}), \hat{u} \rangle = 0$. Recalling the non-degeneracy condition (10), we find that $\mu = 0$. Whence, from (12) we get $\hat{r}\Phi'(\hat{v}) = \frac{\partial \Psi}{\partial u}(\hat{r}, \hat{u}) = 0$, for $\hat{v} = \hat{r}\hat{u}$, and $\hat{r} \neq 0$. Therefore, $\Phi'(\hat{v}) = 0$, as desired. □

For $r \in \mathbb{R}$, $\Phi(ru) = \frac{t|r|^p}{p}\|u\|_{1,p}^p - \frac{1}{2}r^2 H(u)$. Keeping an eye on the fibering equation (11), we differentiate $\Phi(ru)$, and solve $\partial_r \Phi(ru) = 0$ with respect to r. As a result, we get:

$$t|r|^{p-2}r\|u\|_{1,p}^p - rH(u) = 0,$$

which, as $r \neq 0$, yields:

$$|r| = \left(\frac{t\|u\|_{1,p}^p}{H(u)}\right)^{\frac{1}{2-p}}. \tag{14}$$

Of course, a necessary condition for (14) to make sense is that $H(u)$ be positive. As r depends on u, we may refer to it as $r(u)$. Substituting $r(u)$ back in $\Phi(ru)$ gives us:

$$\hat{\Phi}(u) := \Phi(r(u)u) = \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{t\|u\|_{1,p}^p}{H(u)}\right)^{\frac{2}{2-p}} H(u). \tag{15}$$

A quick check verifies that $\hat{\Phi}$ is 0-homogeneous. By taking the fibering function

$$M(u) := \frac{t\|u\|_{1,p}^p}{H(u)},$$

from equation (15), we obtain the special case:

$$\forall u \in M^{-1}(1): \quad \hat{\Phi}(u) = \left(\frac{1}{p} - \frac{1}{2}\right)H(u). \tag{16}$$

Note that for all $u \in M^{-1}(1)$, we have $\langle H'(u), u\rangle = 2H(u)$. Hence, the fibering function M satisfies the non-degeneracy condition. Indeed:

$$\forall u \in M^{-1}(1): \quad \langle M'(u), u\rangle = t\frac{p\|u\|_{1,p}^p H(u) - \|u\|_{1,p}^p \langle H'(u), u\rangle}{H^2(u)} = p - 2 \neq 0.$$

On $M^{-1}(1)$ we have $H(u) = t\|u\|_{1,p}^p$, which implies that $\hat{\Phi}|_{M^{-1}(1)}$ is non-negative. Hence, we consider the following minimization problem:

$$\inf_{u\in M^{-1}(1)} \hat{\Phi}(u) =: m. \tag{17}$$

We will show that (17) is solvable. Once this goal is achieved, we will present an argument to prove that any minimizer is indeed an eigenfunction corresponding to λ, which of course will be the result we are seeking.

Our first step is to make sure $M^{-1}(1)$ is non-empty.

Lemma 2.2. *The set $M^{-1}(1)$ is non-empty.*

Proof. Remember the condition $\lambda > (1-t)\lambda_1$. From the variational formulation of λ_1, i. e.:

$$\lambda_1 = \inf_{u\in W_0^{1,2}(D)} \frac{\int_D |\nabla u|^2\, dx}{\int_D u^2\, dx},$$

we infer the existence of a function $u \in W_0^{1,2}(D)$ such that:

$$\lambda_1 < \frac{\int_D |\nabla u|^2\, dx}{\int_D u^2\, dx} < \frac{\lambda}{1-t}.$$

This means that $H(u) > 0$, and as a result, $M(u) > 0$. Though u itself may not belong to $M^{-1}(1)$, for $\xi := (M(u))^{\frac{1}{2-p}}$, we have $M(\xi u) = 1$. □

We will have to deal with minimizing sequences for the minimization problem (17). Our next step is proving that these sequences are bounded.

Lemma 2.3. *Minimizing sequences of (17) are bounded in $W_0^{1,2}(D)$.*

Proof. Let us fix a minimizing sequence $(u_n) \subseteq M^{-1}(1)$, and assume that:

$$\lim_{n\to\infty} \hat{\Phi}(u_n) = \lim_{n\to\infty} \left(\frac{1}{p} - \frac{1}{2}\right) H(u_n) = m. \tag{18}$$

Since $M(u_n) = 1$, we have:

$$t \int_D |\nabla u_n|^p \, dx + (1-t) \int_D |\nabla u_n|^2 \, dx - \lambda \int_D u_n^2 \, dx = 0. \tag{19}$$

To seek a contradiction, we assume that (u_n) is not bounded. Hence, passing to a subsequence—still denoted as (u_n)—if necessary, $\lim_{n\to\infty} \|u_n\|_{1,2} = \infty$. From (19), we have $(1-t)\int_D |\nabla u_n|^2 \, dx \leq \lambda \int_D u_n^2 \, dx$. As a result, $\int_D u_n^2 \, dx \to \infty$ as well.

We set $v_n := \frac{u_n}{\|u_n\|_2}$, and keep in mind that $\|v_n\|_2 = 1$. From the last inequality, we get $\int_D |\nabla v_n|^2 \, dx \leq \frac{\lambda}{1-t}$, hence (v_n) is bounded in $W_0^{1,2}(D)$. This implies the existence of $v \in W_0^{1,2}(D)$, and a subsequence of (v_n)—still denoted by (v_n)—such that:

$$\begin{cases} v_n \rightharpoonup v & \text{in } W_0^{1,2}(D), \\ v_n \rightharpoonup v & \text{in } W_0^{1,p}(D), \\ v_n \to v & \text{in } L^2(D). \end{cases}$$

Returning to (19), dividing the entire equation by $\|u_n\|_2^p$ and rearranging the terms, we get $t \int_D |\nabla v_n|^p \, dx = \frac{H(u_n)}{\|u_n\|_2^p}$. The right hand side tends to zero, because of (18) and the fact that $\|u_n\|_2 \to \infty$. Therefore, $\int_D |\nabla v_n|^p \, dx \to 0$.

As $v_n \rightharpoonup v$ in $W_0^{1,p}(D)$, by the weak lower semi-continuity of norms we infer that $\|v\|_{1,p} \leq \liminf_{n\to\infty} \|v_n\|_{1,p} = 0$. Thus, $v = 0$, and as a consequence $v_n \to 0$ in $L^2(D)$. However, this implies that $\|v_n\|_2 \to 0$, which is impossible, since $\|v_n\|_2 = 1$ for every n. So, the minimizing sequence (u_n) must be bounded. □

Lemma 2.4. *The minimization problem (17) has a non-zero solution.*

Proof. We consider a minimizing sequence (u_n), and assume, for the time being, that $m \neq 0$. By Lemma 2.3, (u_n) is bounded in $W_0^{1,2}(D)$. So, there is a subsequence—still denoted (u_n)—and a function $u \in W_0^{1,2}(D)$, such that:

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,2}(D), \\ u_n \rightharpoonup u & \text{in } W_0^{1,p}(D), \\ u_n \to u & \text{in } L^2(D). \end{cases} \tag{20}$$

Note that since $M(u_n) = 1$, equation (19) still holds, which together with (20) implies:

$$t \int_D |\nabla u|^p \, dx + (1-t) \int_D |\nabla u|^2 \, dx - \lambda \int_D u^2 \, dx \leq 0. \tag{21}$$

Let us show that $H(u) \neq 0$. Assuming the contrary, (20) leads to $\limsup_{n\to\infty} H(u_n) \leq H(u) = 0$. Since $H(u_n) \geq 0$, we infer $\limsup_{n\to\infty} H(u_n) = 0$. Whence, we can extract a subsequence—still denoted (u_n)— such that $\lim_{n\to\infty} H(u_n) = 0$. This, in turn, implies $\lim_{n\to\infty} \hat{\Phi}(u_n) = 0$, so $m = 0$. This is a contradiction, hence $H(u) \neq 0$.

From (21), we get $M(u) \leq 1$. On the other hand, since $(u_n) \subseteq M^{-1}(1)$:

$$\hat{\Phi}(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) t \int_D |\nabla u_n|^p \, dx.$$

This, in turn, implies that $\hat{\Phi}(u) \leq \liminf_{n\to\infty} \hat{\Phi}(u_n) = m$. Thus, for u to be a solution of (17), we need to show that in fact $M(u) = 1$. We already know that $M(u) \leq 1$. Hence, in order to derive a contradiction let us assume that $M(u) < 1$. Then, by setting $\xi = (M(u))^{\frac{1}{2-p}}$, we would get $M(\xi u) = 1$. As $\xi < 1$, we have:

$$\hat{\Phi}(\xi u) = \xi^2 \hat{\Phi}(u) \leq \xi^2 m < m,$$

where in the last inequality we have used the assumption $m \neq 0$. This is a contradiction.

So, to complete the proof we need to show that $m \neq 0$. Again, we seek a contradiction and assume that $m = 0$. Let (u_n) be a minimizing sequence. Then:

$$\hat{\Phi}(u_n) = \left(\frac{1}{p} - \frac{1}{2}\right) t \int_D |\nabla u_n|^p \, dx \to 0. \tag{22}$$

Since (u_n) is bounded in $W_0^{1,2}(D)$, it contains a subsequence—still denoted as (u_n)—such that:

$$\begin{cases} u_n \rightharpoonup u & \text{in } W_0^{1,2}(D), \\ u_n \rightharpoonup u & \text{in } W_0^{1,p}(D), \\ u_n \to u & \text{in } L^2(D), \end{cases} \tag{23}$$

for some $u \in W_0^{1,2}(D)$. From (21), (23), and the weak lower semi-continuity of the $W_0^{1,2}(D)$-norm we deduce that $u = 0$. Next, we set $v_n = \frac{u_n}{\|u_n\|_2}$, and remember that $\|v_n\|_2 = 1$. On the other hand, since $H(u_n) > 0$, we see that the sequence (v_n) is bounded in $W_0^{1,2}(D)$. Whence, it contains a subsequence—still denoted as (v_n)—such that:

$$\begin{cases} v_n \rightharpoonup v & \text{in } W_0^{1,2}(D), \\ v_n \rightharpoonup v & \text{in } W_0^{1,p}(D), \\ v_n \to v & \text{in } L^2(D), \end{cases} \tag{24}$$

for some $v \in W_0^{1,2}(D)$. Note that since $(u_n) \subseteq M^{-1}(1)$, we have:

$$t \int_D |\nabla u_n|^p \, dx + (1-t) \int_D |\nabla u_n|^2 \, dx - \lambda \int_D u_n^2 \, dx = 0. \tag{25}$$

Dividing equation (25) by $\|u_n\|_2^2$ yields:

$$t\|u_n\|_2^{p-2} \int_D |\nabla v_n|^p \, dx + (1-t) \int_D |\nabla v_n|^2 \, dx - \lambda = 0. \tag{26}$$

Since $(v_n) \subseteq W_0^{1,2}(D)$ is bounded, (26) implies $\int_D |\nabla v_n|^p \, dx \to 0$. This, in conjunction with the Poincaré inequality, implies that $v = 0$. As a result, from (24) we infer that $v_n \to 0$ in $L^2(D)$. This is a contradiction since $\|v_n\|_2 = 1$. □

Completing the proof of Theorem 1.1 for $1 < p < 2$: We achieve this by applying Theorem 2.1. Let $r(u)$ be as in (14), and set $\hat{\Phi}(u) := \Phi(r(u)u)$. The fibering function is $M(u) := \frac{t\|u\|_{1,p}^p}{H(u)}$, where $H(u) = \lambda \int_D u^2 \, dx - (1 - t) \int_D |\nabla u|^2 \, dx$. As we have seen, the minimization problem

$$\inf_{u \in M^{-1}(1)} \hat{\Phi}(u)$$

is solvable. So, if $\hat{u} \in M^{-1}(1)$ is a solution, then according to Theorem 2.1, $\hat{v} := \hat{r}\hat{u}$ (with $\hat{r} = r(\hat{u}) = 1$) is a critical point of Φ. □

Remark 2.1. Note that since the minimizing sequence (u_n) can be assumed to be non-negative, the critical point $\hat{u}$ turns out to be non-negative as well. As a result, by an application of the strong maximum principle (see, e. g., [26]) we deduce that $\hat{u}$ is in fact strictly positive.

3 A priori bounds, regularity, and proofs of theorems 1.2 and 1.3

Remember that the appropriate spaces for solutions of (6) are $W_0^{1,2}(D)$ if $p \in (1,2)$, and $W_0^{1,p}(D)$ when $p > 2$. The solution is the function u satisfying:

$$t \int_D |\nabla u|^{p-2} \nabla u \cdot \nabla v \, dx + (1-t) \int_D \nabla u \cdot \nabla v \, dx = \lambda \int_D uv \, dx, \tag{27}$$

for any $v \in W_0^{1,2}(D)$ when $p \in (1,2)$, and for any $v \in W_0^{1,p}(D)$ when $p > 2$. As is clear from Theorem 1.1, equation (6) has a non trivial solution when $\lambda > (1-t)\lambda_1$. In what follows, we derive bounds for such solutions. First, define $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$.

Note that the differential operator in (6) is a non-degenerate elliptic operator if $p \geq 2$ and a singular one if $p \in (1,2)$. Thus, we may employ established results to derive bounds [9, 15, 16]. We begin with some a priori supremum bound that holds for solutions of (6).

Lemma 3.1 (**Supremum Bound**). *Let $1 < p < \infty$, and assume that u solves (6). Then, there exists a constant $C = C(t, \lambda, n, D)$ such that:*

$$\sup_D |u| \leq C \|u\|_{L^2(D)}.$$

Moreover, if $0 < a < b < \infty$, then $0 < \inf_{a<\lambda<b} C \leq \sup_{a<\lambda<b} C < \infty$.

Proof. The proof is a slight adaptation of the proof of Theorem 8.15 in [15]. Hence, we provide only a brief outline. Since both u and $-u$ are solutions, it is enough to show the bound for u^+. Also, if u solves (6), then $u \in W_0^{1,2}(D)$ for $1 < p < \infty$.

Let $N > 0$ and $\beta \geq 1$ and consider the functions $H \in C^1([0,\infty))$ of the form:

$$H(z) := \begin{cases} z^\beta, & \text{if } z \in [0, N), \\ \text{linear}, & \text{if } z \geq N. \end{cases}$$

The proof relies on the use of the test function

$$v = G(u^+) := \int_0^{u^+} |H'(s)|^2 ds.$$

In general, $v \in W_0^{1,2}(D)$, and in particular, if $p \geq 2$, then $v \in W_0^{1,p}(D)$. Using v in (27), we obtain:

$$\int_D \left(t|\nabla u^+|^p + (1-t)|\nabla u^+|^2 \right) G'(u^+) \, dx = \lambda \int_D uv \leq \lambda \int_D G'(u^+)(u^+)^2 \, dx,$$

since $G(s) \leq sG'(s)$. Disregarding the first term on the left hand side, we obtain:

$$\int_D |\nabla H(u^+)|^2 \, dx \leq \frac{\lambda}{1-t} \int_D \left(H'(u^+)u^+ \right)^2 \, dx.$$

The rest of the proof uses the Sobolev embedding and the Moser iteration to obtain the conclusion. The last part of the statement follows by keeping track of the constants in the proof. □

Next, we discuss bounds for solutions. We begin with the case $p > 2$. From here on, λ_1 denotes the first eigenvalue of $-\Delta$, and λ_p denotes the first eigenvalue of $-\Delta_p$, on D. We state the following Poincaré inequalities:

$$\begin{cases} \lambda_1 \int_D u^2 \, dx \leq \int_D |\nabla u|^2 \, dx, & \forall u \in W_0^{1,2}(D), \\ \lambda_p \int_D |u|^p \, dx \leq \int_D |\nabla u|^p \, dx, & \forall u \in W_0^{1,p}(D). \end{cases} \tag{28}$$

Now, we present the proofs of theorems 1.2 and 1.3:

Proof. **(Theorem 1.2)**

(i) We recall Lemma 3.1 and the results in [9, 16]. Thus, $u \in C^{1,\alpha}_{loc} \cap L^\infty(D)$. Since $p > 2$, the operator in (6) is uniformly elliptic. Applying the Schauder estimates implies that $u \in C^{2,\alpha}_{loc}(D)$. In the set $D \setminus \{\nabla u = 0\}$, one may differentiate and bootstrap to conclude $u \in C^\infty$.

(ii) Using u as a test function in (27), we obtain

$$t \int_D |\nabla u|^p \, dx + (1-t) \int_D |\nabla u|^2 \, dx = \lambda \int_D u^2 \, dx. \tag{29}$$

We use the Poincaré inequality (28) on the right hand side of (29) to obtain:

$$\int_D |\nabla u|^p \, dx \leq \left(\frac{\lambda - (1-t)\lambda_1}{\lambda_1 t} \right) \int_D |\nabla u|^2 \, dx. \tag{30}$$

Applying the Hölder inequality, we get:

$$\int_D |\nabla u|^2 \, dx \leq \left(\int_D |\nabla u|^p \, dx \right)^{2/p} |D|^{(p-2)/p}.$$

Thus, (30) leads to:

$$\int_D |\nabla u|^p \, dx \leq \left(\frac{\lambda - (1-t)\lambda_1}{\lambda_1 t} \right)^{p/(p-2)} |D|. \tag{31}$$

The result follows by applying the Poincaré inequality in (28).

(iii) We apply Lemma 3.1, the Poincaré inequality (28), the Hölder inequality, and (31) to obtain:

$$\begin{aligned} \sup_D |u| &\leq C \left(\int_D u^2 \, dx \right)^{1/2} \\ &\leq \frac{C}{\lambda_1^{1/2}} \left(\int_D |\nabla u|^2 \, dx \right)^{1/2} \\ &\leq \frac{C}{\lambda_1^{1/2}} \left(\int_D |\nabla u|^p \, dx \right)^{1/p} |D|^{(p-2)/2p} \\ &\leq \frac{C}{\lambda_1^{1/2}} \left(\frac{\lambda - (1-t)\lambda_1}{\lambda_1 t} \right)^{1/(p-2)} |D|^{1/2}. \end{aligned}$$

which completes the proof. □

Proof. **(Theorem 1.3)**

(i) We recall Lemma 3.1 and [9, 16]. Thus, $u \in C^{1,\alpha}_{loc} \cap L^\infty(D)$. Since $p < 2$, the operator is uniformly elliptic in the set $D \setminus \{\nabla u = 0\}$. Applying the Schauder estimates, we obtain that $u \in C^{2,\alpha}_{loc}$. Once again by differentiating and bootstrapping we obtain $u \in C^\infty$ in $D \setminus \{\nabla u = 0\}$.

(ii) Use the test function u in (27) and apply the Poincaré inequalities (28) to obtain:

$$\lambda_p \int_D |u|^p \, dx \leq \int_D |\nabla u|^p \, dx \leq \frac{\lambda - \lambda_1(1-t)}{t} \int_D u^2 \, dx. \tag{32}$$

Set $M := \sup_D |u|$. From Lemma 3.1, for some $C = C(\lambda, t, n, D)$, we note that $M^2 \leq C^2 M^{2-p} \int_D |u|^p \, dx$, implying that $M \leq C^{2/p} \|u\|_{L^p(D)}$. Next, by using this inequality, for an appropriate $C_1 = C_1(C, p)$, we get:

$$\begin{aligned} \int_D u^2 \, dx &\leq C M^{2-p} \int_D |u|^p \, dx \\ &\leq C_1 \left(\int_D |u|^p \, dx \right) \left(\int_D |u|^p \, dx \right)^{(2-p)/p} \\ &\leq C_1 \left(\int_D |u|^p \, dx \right)^{2/p}. \end{aligned} \tag{33}$$

By using (33) in (32) we obtain:

$$\lambda_p \int_D |u|^p \, dx \leq C_1 \left(\frac{\lambda - \lambda_1(1-t)}{t} \right) \left(\int_D |u|^p \, dx \right)^{2/p}.$$

After simplification, one gets:

$$\int_D |u|^p \, dx \geq C_2 \left(\frac{t \lambda_p}{\lambda - \lambda_1(1-t)} \right)^{p/(2-p)},$$

in which $C_2 = C_2(\lambda, n, p, D)$. Lemma 3.1 implies that $C_2 > 0$ if $\lambda > 0$.

(iii) Follows from:

$$(\sup_D |u|) \, |D|^{1/p} \geq \|u\|_{L^p(D)}.$$

□

References

[1] Aizicovici, S., Papageorgiou, N. S., and Staicu, V. Nodal solutions for $(p, 2)$-equations. *Trans. Amer. Math. Soc.*, 367(10):7343–7372, 2015.

[2] Aris, R. *Mathematical modelling techniques*, volume 24 of *Research notes in mathematics*. Pitman, 1978.

[3] Bartolo, R., Candela, A., and Salvatore, A. On a class of superlinear (p, q)-Laplacian type equations on $\mathbb{R}^N$. *J. Math. Anal. Appl.*, 438(1):29–41, 2016.

[4] Benci, V., D'Avenia, P., Fortunato, D., and Pisani, L. Solitons in several space dimensions: Derrick's problem and infinitely many solutions. *Arch. Ration. Mech. Anal.*, 154(4):297–324, Oct 2000.

[5] Chaves, M. F., Ercole, G., and Miyagaki, O. H. Existence of a nontrivial solution for a (p, q)-Laplacian equation with p-critical exponent in $\mathbb{R}^N$. *Bound. Value Probl.*, 236(1):15, 2014.

[6] Chaves, M. F., Ercole, G., and Miyagaki, O. H. Existence of a nontrivial solution for the (p, q)-Laplacian in $\mathbb{R}^N$ without the Ambrosetti-Rabinowitz condition. *Nonlinear Analysis*, 114:133–141, 2015.

[7] Chen, C., Bao, J., and Song, H. Multiple solutions for a class of fractional (p, q)-Laplacian system in $\mathbb{R}^N$. *J. Math. Phys.*, 59(3):031505, 2018.

[8] Cherfils, L. and Il'yasov, Y. On the stationary solutions of generalized reaction diffusion equations with $p\&q$-Laplacian. *Commun. Pure Appl. Anal.*, 4(1):9–22, 2005.

[9] DiBenedetto, E. {C1} + local regularity of weak solutions of degenerate elliptic equations. *Nonlinear Anal.*, 7(8):827–850, 1983.

[10] Fărcăşeanu, M., Mihăilescu, M., and Stancu-Dumitru, D. On the set of eigenvalues of some PDEs with homogeneous Neumann boundary condition. *Nonlinear Anal.*, 116(0):19–25, 2015.

[11] Fife, P. C. *Mathematical Aspects of Reacting and Diffusing Systems*. Lecture Notes in Biomathematics. Springer, 1979.

[12] Figueiredo, G. M. Existence of positive solutions for a class of $p\&q$ elliptic problems with critical growth on $\mathbb{R}^N$. *J. Math. Anal. Appl.*, 378(2):507–518, 2011.

[13] Figueiredo, G. M. Existence and multiplicity of solutions for a class of $p\&q$ elliptic problems with critical exponent. *Math. Nachr.*, 286(11-12):1129–1141, 2013.

[14] Gasiński, L. and Papageorgiou, N. S. Asymmetric $(p,2)$-equations with double resonance. *Calc. Var. Partial Differential Equations*, 56(3):88, 2017.

[15] Gilbarg, D. and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, 2001.

[16] Ladyzhenskaya, O. A. and Uraltseva, N. N. *Linear and Quasilinear Elliptic Equations*. Academic Press, 1968.

[17] Lv, Y. and Ou, Z.-Q. Existence of weak solutions for a class of (p,q)-Laplacian systems. *Bound. Value Probl.*, 2017(1):1–16, 2017.

[18] Mercuri, C. and Squassina, M. Global compactness for a class of quasi-linear elliptic problems. *Manuscripta Math.*, 140(1):119–144, 2013.

[19] Mihailescu, M. An eigenvalue problem possessing a continuous family of eigenvalues plus an isolated eigenvalue. *Commun. Pure Appl. Anal.*, 10:701–708, 2011.

[20] Papageorgiou, N. S. and Rădulescu, V. D. Qualitative phenomena for some classes of quasilinear elliptic equations with multiple resonance. *Appl. Math. Optim.*, 69(3):393–430, Jun 2014.

[21] Papageorgiou, N. S. and Rădulescu, V. D. Noncoercive resonant $(p,2)$-equations. *Appl. Math. Optim.*, 76(3):621–639, 2017.

[22] Papageorgiou, N. S., Rădulescu, V. D., and Repovš, D. D. On a class of parametric $(p,2)$-equations. *Appl. Math. Optim.*, 75(2):193–228, 2017.

[23] Papageorgiou, N. S., Rădulescu, V. D., and Repovš, D. D. Existence and multiplicity of solutions for resonant $(p,2)$-equations. *Adv. Nonlinear Stud.*, 18(1):105–129, 2017.

[24] Pohozaev, S. I. The fibering method and its applications to nonlinear boundary value problem. *Rend. Istit. Mat. Univ. Trieste*, 31(1–2):235–305, 1999.

[25] Radulescu, V. and Chorfi, N. Continuous spectrum for some classes of $(p,2)$-equations with linear or sublinear growth. *Miskolc Math. Notes*, 17(2):817–826, 2016.

[26] Serrin, J. On the strong maximum principle for quasilinear second order differential inequalities. *J. Funct. Anal.*, 5(2):184–193, 1970.

[27] Struwe, M. *Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems*. Springer, fourth edition, 2008.

[28] Vélin, J. On an existence result for a class of (p,q)-gradient elliptic systems via a fibering method. *Nonlinear Anal.*, 75(16):6009–6033, 2012.

[29] Vélin, J. Multiple solutions for a class of (p,q)-gradient elliptic systems via a fibering method. *Proc. Roy. Soc. Edinburgh Sect. A*, 144:363–393, 2014.

[30] Wu, M. and Yang, Z. Positive solutions for a class of p-q-Laplacian type equations with potentials in $\mathbb{R}^N$. *J. Abstr. Differ. Equ. Appl.*, 2(2):93–109, 2012.

[31] Yin, H. and Yang, Z. A class of p-q-Laplacian type equation with concave-convex nonlinearities in bounded domain. *J. Math. Anal. Appl.*, 382(2):843–855, 2011.

[32] Yin, H. and Yang, Z. Multiplicity of positive solutions to a p-q-Laplacian equation involving critical nonlinearity. *Nonlinear Anal.*, 75(6):3021–3035, 2012.