NON-LOCAL OPTIMAL REARRANGEMENT PROBLEMS

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Abstract

The present thesis is a result of research done in the field of optimization problems related to non-local partial differential equations in past three years. More precisely it focuses on optimal rearrangement problems in the context of (non-local) fractional Laplace operators.

In various problems in physics, fluid mechanics and economics certain functions belong to the same rearrangement class (see Section 1.1). Then one is interested in maximization or minimization of particular energies and analyzing the properties of the corresponding optimal solutions. This field of mathematics is based on research of Geoffrey Burton and his collaborators and students in 80s and 90s (see [13], [16], [17], [14], [15]). Later the results have been generalized for the $p-$Laplace operators (see [43], [33], [42], [31]), biharmonic operator (see [35], [22]), as well as constrained cases (see [44], [32]).

Most of the classical results known for the Laplace operators have been generalized in the non-local setting for the first time in this work. Same time we show that the non-locality implies new phenomena not observed for local operators. Similarly to the results obtained in [44], the solution of the energy minimization problem we obtain is not a characteristic function (a bang-bang function).

Another important feature is the connection between optimal rearrangement problems and free boundary problems, particularly the obstacle problem, known for the classical (local) optimal rearrangement problems. Our analysis allows to derive the fractional version of the so-called normalized obstacle problem from the rearrangement context, and obtain a new type of equation for its solutions. The results in Chapter 3 and Chapter 4 of this thesis are published in [8] and [7] respectively.

Short about the structure of the thesis: in Chapter 1 we introduce some backgrounds and applications; in Chapter 2 we introduce some preliminaries in fractional setting; in Chapter 3, we study the fractional optimal maximization problem; in Chapter 4, we study the fractional optimal minimization problem; In Chapter 5, we study the fractional analogue of the variational minimization problem introduced in [37].

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Contents

Chapter 1

Introduction

1.1 The Rearrangement Problem

In many applications (see Section 1.2) certain quantities are not given, but one has information about the distribution of their values. This leads to the mathematical theory of optimal rearrangement problems. We say that two real valued functions f and g defined on a bounded domain $\Omega \subset \mathbb{R}^n$ have the same value distribution, and write $f \sim g$, if

$$\mathcal{L}^n\left(f^{-1}\left([\beta,\infty)\right)\right) = \mathcal{L}^n\left(g^{-1}\left([\beta,\infty)\right)\right) \quad \text{for any real } \beta.$$

One can easily see that $\sim$ defined above is an equivalence relation, so one can define the rearrangement class generated by a function f_0 as follows

$$\mathcal{R}_{f_0} = \{f; f \sim f_0\}.$$

The function f_0 is then called the generator of the rearrangement class $\mathcal{R}_{f_0}$. In present work we will take f_0 from $L^\infty(\Omega)$ and denote by $\bar{\mathcal{R}}_{f_0}$ the weak* closure of $\mathcal{R}_{f_0}$ in $L^\infty(\Omega)$.

The following optimal rearrangement problems have been discussed in [13], [16] and [17]. Let u_f be the solution to the Dirichlet problem

$$\begin{cases} -\Delta u_f(x) = f(x) & \text{in } \Omega, \\ u_f(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where f is the source function and u_f is the potential. One can think of the external heating and heat distribution respectively. One is interested in the minimization and maximization of the energy functional

$$E(f) = \frac{1}{2}\int_\Omega |\nabla u_f|^2 dx$$

for a given distribution of external heating, i.e., the minimization/maximization of $E(f)$ over the set $\mathcal{R}_{f_0}$. In this thesis, we will restrict ourselves to the case of the rearrangement classes generated by characteristic functions $f_0 = \chi_{E_0}$, i.e.,

$$\mathcal{R}_\beta = \{f : f = \chi_E, |E| = \beta\}, \tag{1.2}$$

where $\beta = |E_0| \in (0, |\Omega|)$.

Since the set $\mathcal{R}_\beta$ is not closed in weak$^*-$topology, one has to consider the problem (1.1) in weak* closure of $\mathcal{R}_\beta$, which is denoted by $\bar{\mathcal{R}}_\beta$.

The convexity and and sequentially compactness of $\bar{\mathcal{R}}_\beta$ in weak$^*-$topology follows from Lemma 2.2 in [16], which we present below in notations we use, and in the setting of $p = 1,\ q = \infty$.

Theorem 1.1.1. *Let (Ω, μ) be a measure space, $f_0 \in L^1(\mu)$ and $\mathcal{R}_{f_0}$ be the class of rearrangements of f_0 on Ω. Also, let $\bar{\mathcal{R}}_{f_0}$ denote the closure of $\mathcal{R}_{f_0}$ in the $L^\infty-$topology on $L^\infty(\mu)$. Then $\bar{\mathcal{R}}_{f_0}$ is convex, so $\bar{\mathcal{R}}_{f_0}$ equals the closed convex hull of $\mathcal{R}_{f_0}$. Moreover, $\bar{\mathcal{R}}_{f_0}$ is sequentially compact in the $L^\infty-$topology.*

The following theorem gives the explicit formulation of $\bar{\mathcal{R}}_\beta$ in the case $f_0 = \chi_E$; $|E| = \beta$ (see [17]).

Theorem 1.1.2. *We have the following characterization of $\bar{\mathcal{R}}_\beta$*

$$\bar{\mathcal{R}}_\beta = \left\{ f : 0 \leq f \leq 1,\ \int_\Omega f dx = \beta \right\} \subset L^\infty(\Omega). \tag{1.3}$$

1.2 Application to Fluid Dynamics

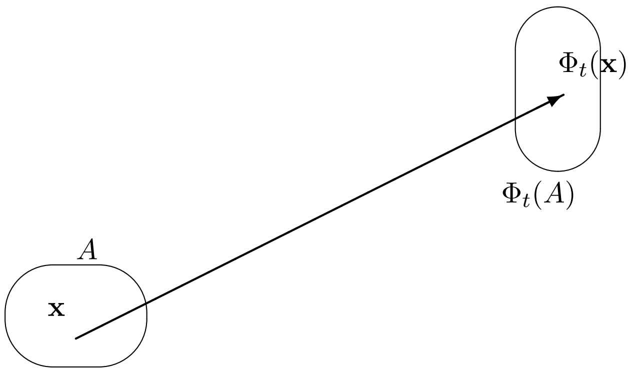
In this section, we will show how rearrangement problems relate to fluid dynamics. The content is based on any standard textbooks such as [1] or [4].

Let Ω be a open, bounded, simply connected domain in $\mathbb{R}^2$ with C^2 boundary. We consider a two-dimensional incompressible flow in $\Omega \times (z_1, z_2)$. Since there is no z-dependence, we denote a two-dimensional velocity field by

$$\mathbf{u}(x, y, z, t) = \mathbf{u}(x, y, t) = (u_1(x, y, t), u_2(x, y, t), 0).$$

There exists a measure-preserving diffeomorphism $\Phi_t : \Omega \to \Omega$ which carries a material point x at time $t = 0$ to the point $\Phi_t(x)$, it will flow to after time t. Since the flow is incompressible, we have

$$|A| = |\Phi_t(A)|, \quad \text{for any volume } A \subset \Omega \text{ and } t > 0.$$



We denote the vorticity of the flow by

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ u_1 & u_2 & 0 \end{vmatrix}, \tag{1.4}$$

A flow is irrotational if $\omega = 0$. For an incompressible fluid, the density of a material element does not change following the flow and by conservation of mass, incompressibility is equivalent to

$$\nabla \cdot \mathbf{u} = 0. \tag{1.5}$$

Since the third entry of u vanishes, (1.5) gives the identity

$$\frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0.$$

This property opens door for streamfunctions $\varphi(x, y, t)$, which is defined by

$$\mathbf{u} = \nabla^{\perp}\phi = (\varphi_y, -\varphi_x, 0). \tag{1.6}$$

Substituting (1.6) in (1.4) we obtain the following relation between vorticity and streamfunction.

$$\boldsymbol{\omega} = (0, 0, -\Delta\varphi). \tag{1.7}$$

The flow is irrotational if its streamfunction is harmonic. It is easy to show that φ is constant on streamlines and hence the following Proposition implies that φ is constant on $\partial\Omega$.

Proposition 1.2.1. *Stationary rigid boundaries are streamlines.*

Proof. Since a flow cannot penetrate a stationary rigid boundary (no inflow and no outflow), $\mathbf{u}$ is perpendicular to the outward normal vector on the boundary. Hence, $\mathbf{u}$ must be locally parallel to the boundary. But $\mathbf{u}$ is locally parallel to streamlines, and the result follows. □

Deduced from Euler's equations, we have the following vorticity equation,

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u}, \tag{1.8}$$

where $\frac{D}{Dt}$ is the material derivative. The equation (1.8) gives the rate of change of vorticity following a fluid particle in a flow. For our 2D flow, we have $\boldsymbol{\omega} = (0, 0, \omega)$ and thus

$$\frac{D\boldsymbol{\omega}}{Dt} = (\boldsymbol{\omega} \cdot \nabla)\,\mathbf{u} = 0. \tag{1.9}$$

This implies that $\boldsymbol{\omega}$ cannot be created or destroyed by the flow. Now, we fix a fluid particle $\mathbf{x}$ at $t = 0$ and observe its vorticity at time t: $\boldsymbol{\omega}_t(\mathbf{x}) = \boldsymbol{\omega}(\Phi_t(x))$. From (1.9), we obtain

$$\boldsymbol{\omega}_t(\mathbf{x}) = \Phi_t \circ \boldsymbol{\omega}_0(\mathbf{x}), \tag{1.10}$$

for any $\mathbf{x} \in \Omega$ and $t > 0$. This implies the following lemma.

Lemma 1.2.1. *The functions $\boldsymbol{\omega}_s(\mathbf{x})$ belong to the same rearrangement class.*

By (1.7), (1.10) and Proposition 1.2.1, we have that the streamfunction $\varphi(x, y, t)$ satisfies the following PDEs,

$$\begin{cases} -\Delta\varphi = \omega & \text{in } \Omega, \\ \varphi = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.11}$$

Thus finding the maximum or minimum of the kinetic energy

$$E(\boldsymbol{u}) = \int_\Omega |\boldsymbol{u}|^2 dx = \int_\Omega |\nabla\varphi|^2\, dx$$

is equivalent to solving the optimal rearrangement problem introduced in Section 1.1.

1.3 A Probabilistic Motivation to Fractional Laplacian

In this section, we would like to introduce the fractional Laplace operaters $(-\Delta)^s$ (see Definition 2.2.2) and fractional Laplacian equations from the probabilistic point of view, which is closer to the physical concept of diffusion and therefore closer to certain applications of rearrangement problems (see [2], [44]).

According to [12], one may consider two examples where probabilistic models motivate fractional Laplacian equations. We consider a function $u : \mathbb{R}^n \to \mathbb{R}$, and a fractional parameter $s \in (0, 1)$. Firstly, we are going to construct a probabilistic process model

Let $\mathbb{N}$ denote positive natural numbers and $I \subset \mathbb{N}$. The probability measure of I is defined by

$$P(I) = c(s) \sum_{k \in I} \frac{1}{|k|^{1+2s}},$$

where $c(s) := \left(\Sigma_{k \in \mathbb{N}} \frac{1}{|k|^{1+2s}} \right)^{-1}$ is the normalization constant, and thus we have $P(\mathbb{N}) = 1$. Also, we denote the unit ball centred at origin by B_1.

Now, we consider a particle jumping randomly in $\mathbb{R}^n$ with discrete time and distance. We denote the time step and distance step by τ and h respectively and set

$$\tau = h^{2s}.$$

We denote the probability of the particle being found at time t in the point x by $u(x,t)$. At each time step, the particle moves in the direction $\nu \in \partial B_1$ according to uniform distribution. It moves $k \in \mathbb{N}$ units of length h, where k is distributed according to the probability law P. Observe that the probability of a jump is inversely proportional to its length, so long jumps are less probable.

We denote the probability of finding a particle at x_0 at time $t+\tau$ by $u(x_0, t+\tau)$, which is the sum of the probabilities of finding the particle at the point $x_0 + kh\nu$ at time t times the probability of having selected such a direction (ν) and such a distance (kh). Hence we have the following expression where τ does not appear on the right hand side.

$$u(x_0, t+\tau) = \frac{1}{|\partial B_1|} \int_{\partial B_1} \left(c(s) \sum_{k \in \mathbb{N}^*} \frac{1}{|k|^{1+2s}} u(x_0 + kh\nu, t) \right) d\mathcal{H}^{n-1}(\nu),$$

where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure and $c(s)/|\partial B|$ is a normalization constant. We subtract $u(x_0, t)$ and obtain,

$$\begin{aligned} &u(x_0, t+\tau) - u(x_0, t) = \\ &\frac{c(s)}{|\partial B_1|} \sum_{k \in \mathbb{N}^*} \int_{\partial B} \frac{u(x_0 + kh\nu, t) - u(x_0, t)}{|k|^{1+2s}} d\mathcal{H}^{n-1}(\nu). \end{aligned}$$

By symmetry, it is easy to see that the equality above remains true if we change ν to $-\nu$. Then, we sum them up and obtain,

$$\begin{aligned} &u(x_0, t+\tau) - u(x_0, t) \\ = \; &\frac{c(s)}{2|\partial B_1|} \sum_{k \in \mathbb{N}^*} \int_{\partial B} \frac{u(x_0 + kh\nu, t) + u(x_0 - kh\nu, t) - 2u(x_0, t)}{|k|^{1+2s}} \\ &d\mathcal{H}^{n-1}(\nu). \end{aligned} \tag{1.12}$$

Now, we divide (1.12) by $\tau = h^{2s}$ and take a limit $h \to 0^+$.

$$\begin{aligned}
&\partial_t u(x,t) = \lim_{\tau\to 0} \frac{(x_0, t+\tau) - u(x_0,t)}{\tau} \\
= \ & \lim_{h\to 0} \frac{c(s)h}{2|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x_0+kh\nu,t)+u(x_0-kh\nu,t)-2u(x_0,t)}{|hk|^{1+2s}} \\
& d\mathcal{H}^{n-1}(\nu) \\
= \ & \frac{c(s)}{2|\partial B_1|} \int_{\mathbb{R}} \int_{\partial B_1} \frac{u(x_0+r\nu,t)+u(x_0-r\nu,t)-2u(x_0,t)}{|r|^{1+2s}} \\
& d\mathcal{H}^{n-1}(\nu)dr \\
= \ & \frac{c(s)}{2|\partial B_1|} \int_{\mathbb{R}^n} \frac{u(x_0+y,t)+u(x_0-y,t)-2u(x_0,t)}{|y|^{n+2s}} dy \\
= \ & -c(n,s)(-\Delta)_x^s u(x_0,t),
\end{aligned} \tag{1.13}$$

where in the third equality we apply the Riemann sum and $c(n,s)$ is a suitable constant. We reach

$$\partial_t u + (-\Delta)^s u = 0 \tag{1.14}$$

as required (see Definition 2.2.2, Lemma 2.2.1).

We keep the probability model above, and set a subdomain Ω in $\mathbb{R}^n$. New value will be assigned to a particle according to the rule consisting of two cases. In one case when the particle jumps to a point in $\mathbb{R}^n \setminus \Omega$, it will be assigned a prescribed value u_0 (this new value will replace the old one). In the other case when the particle jumps to a point in Ω, no new value will be assigned. In this situation, we obviously have $u(x) = u_0(x)$ for $x \in \mathbb{R}^n \setminus \Omega$. Then, our question is that what is the expected value of the particle which starts its motion at $x_0 \in \Omega$. Indeed, the expected value at x_0 is the average of all values at $x_0 + kh\nu$, weighted by the probability of jumps with the parameters h and ν. Therefore, we obtain,

$$u(x_0) = \frac{c(s)}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x_0+kh\nu)}{|k|^{1+2s}} d\mathcal{H}^{n-1}(\nu).$$

By changing ν to $-\nu$, we have

$$u(x_0) = \frac{c(s)}{|\partial B_1|} \sum_{k\in\mathbb{N}^*} \int_{\partial B_1} \frac{u(x_0-kh\nu)}{|k|^{1+2s}} d\mathcal{H}^{n-1}(\nu).$$

By summing them up, we have,

$$2u(x_0) = \frac{c(s)}{|\partial B_1|} \sum_{k\in\mathbb{N}} \int_{\partial B_1} \frac{u(x_0+kh\nu)+u(x_0-kh\nu)}{|k|^{1+2s}} d\mathcal{H}^{n-1}(\nu).$$

Since $c(s)/|\partial B_1|$ is a normalizing constant, we subtract $2u(x_0)$ and obtain,

$$0 = \frac{c(s)}{|\partial B_1|} \sum_{k\in\mathbb{N}} \int_{\partial B_1} \frac{u(x_0 + kh\nu) + u(x_0 - kh\nu) - 2u(x_0)}{|k|^{1+2s}} d\mathcal{H}^{n-1}(\nu).$$

Now, we adopt the similar techniques in (1.13) and take a limit $h \to 0^+$ to obtain the following fractional boundary value equations,

$$\begin{cases} (-\Delta)^s u = 0, & \text{in } \Omega, \\ u = u_0, & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.15}$$

which is the homogeneous fractional steady-state diffusion equation in a bounded domain.

Classical diffusion occurs in numerous problems which describe diffusion of some energy field in nature, e.g., electrical potential and temperature field. By [51], the normal diffusion is represented by the classical Heat equation or Fokker-Planck linear equation. From above, (1.14) is derived by a random jump and it is called the standard linear evolution partial differential equation involving diffusion and fractional operators. In order to describe anomalous and long-range diffusion, which explains a great variety of phenomena in areas of physics, finance, biology, ecology, geophysics, and etc., studies prefer to use fractional equations instead of classical ones.

1.4 The Obstacle problem

One can derive the classical Dirichlet boundary value problem

$$\begin{cases} -\Delta u = f(x) \text{ in } \Omega, \\ u(x) = \varphi(x) \text{ on } \partial\Omega. \end{cases} \tag{1.16}$$

from the following simple minimization problem,

$$\text{minimize: } \Phi(v) = \int_\Omega \frac{1}{2}|\nabla v|^2 + f(x)v(x)dx$$

over the set

$$K = \left\{ v \in W^{1,2}(\Omega) : v - \varphi \in W_0^{1,2}(\Omega) \right\}.$$

This can be seen as a mathematical model of an elastic membrane attached at the boundary at level φ and subject to an external force f. One can get the classical obstacle problem by adding an obstacle constraint to the problem above, i.e., minimize $\Phi(v)$ over

$$\hat{K}_g = \{u \in K : u \geq g\}.$$

Though it seems to be a very simple constraint, it leads to rather complicated and challenging mathematics (see [18], [36], [47]). The main difficulty is the analysis of the so-called coincidence set $\{x : u(x) = g(x)\}$ and its boundary $\partial\overline{\{x : u(x) > g(x)\}}$, known as **the Free Boundary**.

The case when $f \equiv 1$ and $g \equiv 0$ is called the normalized obstacle problem, which heuristically models a heavy membrane hanging above a rigid plane surface. In this case the functional can be rewritten as

$$J(v) = \int_{\Omega} \frac{1}{2}|\nabla v|^2 + v^+ dx, \tag{1.17}$$

where $v^+ = \max\{0, v\}$. The minimizer of (1.17) is the weak solution to the famous obstacle problem equation

$$\Delta u(x) = \chi_{\{u(x)>0\}},$$

which is the Euler-Lagrange equation for the normalized obstacle problem.

One can prove that the solution is a $C^{1,1}$ function (see [18]). Moreover, the following theorem is true (see [38]).

Theorem 1.4.1. *Let $u(x)$ be a solution to the normalized obstacle problem in Ω. Then there exists a constant $C(n)$ depending only on the dimension n such that if $u(y) = 0$ and $B_{s/4} \subset \Omega$ then*

$$|D^2u(x)| \leq C(n) \qquad \text{for every } x \in B_{s/8}(y) \cap \{u > 0\}.$$

Moreover, $u(x)$ is analytic in $\{u > 0\}$.

The most challenging results however concern the regularity of the free boundary. It has been proven that the free boundary is $C^{1,\alpha}$ up to a singular set $\Sigma(u)$. Moreover, $\Sigma(u)$ is contained in a countable union of C^1-manifolds of dimension $k \leq n-1$ ([18], [36], [47]).

1.5 Connection between Obstacle Problems and Rearrangement Problems

Let Ω be open and bounded in $\mathbb{R}^n$. We consider the functional

$$\Phi(f) := \int_{\Omega} |\nabla u_f|^2 \, dx, \tag{1.18}$$

where u_f is the unique solution of the Dirichlet boundary value problem with $f \in \bar{\mathcal{R}}_\beta$

$$\begin{cases} -\Delta u_f(x) = f(x) \text{ in } \Omega, \\ u_f = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{1.19}$$

The connection between obstacle problems and rearrangement problems is demonstrated by the following maximization problem and minimization problem.

1. We consider the maximization problem

$$\Phi(f) \to \max,$$

over $f \in \bar{\mathcal{R}}_\beta$.

This problem and its variations, such as the minimization problem and its $p-$harmonic and constraint cases, as well as the analogous eigenvalue problems, has been studied by various authors (see [13, 17, 34, 24, 25, 26, 19, 20, 23, 43, 33, 44, 39, 49]). The results, for this particular setting, can be formulated in the following theorem,

Theorem 1.5.1. *There exists a solution $\hat{f} \in \mathcal{R}_\beta$ such that*

$$\Phi(f) \leq \Phi(\hat{f})$$

for any $f \in \bar{\mathcal{R}}_\beta$. Moreover, there exists a constant $\alpha > 0$ such that

$$\hat{f} = \chi_{\{\hat{u} > \alpha\}},$$

where $\hat{u} = u_{\hat{f}}$.

Let us observe that as a result the function $U := \alpha - \hat{u}$ will be a solution of the following unstable obstacle-like problem,

$$-\Delta u = \chi_{\{u>0\}}.$$

2. We consider the minimization problem

$$\Phi(f) \to \min, \tag{1.20}$$

over $f \in \bar{\mathcal{R}}_\beta$.

This minimization problem is related to the stationary heat equation whose time derivative vanishes,

$$\partial_t u - \Delta u(x) = f(x) \text{ in } \Omega. \tag{1.21}$$

The external heat source is modelled by $f(x)$ and the Dirichlet boundary condition, $u = 0$ on $\partial\Omega$, models the constant temperature on the boundary of Ω. The weak solution u_f,which models heat distribution, depends on the force function f. We denote the minimizer of (1.18) $\bar{\mathcal{R}}_\beta$ by $\hat{f}$, which gives the most uniform distribution $u_{\hat{f}}$. The results, for this particular setting, can be formulated in the following theorem,

Theorem 1.5.2. *There exists a unique solution $\hat{f} \in \mathcal{R}_\beta$ such that*

$$\Phi(f) \geq \Phi(\hat{f})$$

for any $f \in \bar{\mathcal{R}}_\beta$. *Also, for the function* $\hat{u} = u_{\hat{f}}$ *there exists a constant* $\alpha > 0$ *such that*

$$\begin{cases} 0 < \hat{u} \leq \alpha \ in\ \Omega, \\ \hat{f} = \chi_{\{\hat{u}<\alpha\}}, \\ \hat{u} = 0 \ in\ \{\hat{f} = 0\}. \end{cases} \tag{1.22}$$

Moreover, the function $\hat{U} = \alpha - \hat{u}$ *is the minimizer of the functional*

$$J(w) = \int_\Omega |\nabla w|^2 + 2\max(w, 0)dx,$$

among functions $w \in W_\alpha := \{w \in W^{1,2}(\Omega) : w = \alpha \ on\ \partial\Omega\}$, *and solves the obstacle problem equation,*

$$\Delta U = \chi_{\{U>0\}}. \tag{1.23}$$

Proof. The existence and uniqueness of the minimizer as well as the properties (1.22) follow from [17, Theorem 2.1]. Let us now show that $\hat{U} = \alpha - \hat{u}$ minimizes J. We introduce the functional

$$I(w) := \int_\Omega |\nabla w|^2 + 2\hat{f}w dx,$$

where $w \in W_\alpha$. From the classical theory, one has that $\hat{U}$ is the unique minimizer of I, and thus

$$I(\hat{U}) \leq I(v) \quad \text{for any } v \in W_\alpha. \tag{1.24}$$

Since $\hat{f} \in \mathcal{R}_\beta$, we have

$$I(v) \leq J(v) \quad \text{for any } v \in W_\alpha. \tag{1.25}$$

Moreover one has

$$\int_\Omega \hat{u}\hat{f}dx \leq \int_\Omega \hat{u}f dx, \quad \text{for any } f \in \bar{\mathcal{R}}_\beta.$$

Consequently, (1.22) implies that

$$\int_\Omega \hat{U}^+ dx = \int_\Omega \hat{U}dx = \int_\Omega \hat{f}\hat{U}dx,$$

which means

$$I(\hat{U}) = J(\hat{U}). \tag{1.26}$$

Now, (1.24), (1.25) and (1.26) imply that

$$J(\hat{U}) = I(\hat{U}) \leq I(v) \leq J(v), \quad \text{for any } v \in W_\alpha.$$

Thus $\hat{U}$ minimizes J.

The equation (1.23) is classical and can be found in [18], [38], and [47]. □

1.6 Variational Optimization Problem

The last problem we would like to discuss in this thesis is the fractional analogue of the following variational optimization problem from [37].

Let Ω be a bounded domain in $\mathbb{R}^n$ where $n = 2, 3$. For a non-negative function f in Ω, let us consider the classical semi-linear Dirichlet boundary value problem

$$\begin{cases} -\Delta u_\omega(x) + \chi_\omega u_\omega(x) = f(x) & \text{in } \Omega, \\ u_\omega(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.27}$$

The following minimization problem,

$$\text{minimize: } J(\omega) = \int_\Omega |\nabla u_\omega|^2 + \chi_\omega u_\omega^2 dx = \int_\Omega f u_\omega dx,$$

over the set

$$\omega \in O_\beta := \{\omega \subset \Omega, |\omega| \leq \beta < |\Omega|\} \tag{1.28}$$

has been considered in [37].

This can be seen as a mathematical model of an elastic body subject to an non-negative loading f, with an unknown subset ω of prescribed volume and stiffness.

According to [37], the functional J is non-increasing with respect to the set inclusion, i.e., if $\omega_1 \subset \omega_2$ then the solutions to (1.27) satisfies $u_{\omega_1} \leq u_{\omega_2}$ and thus we have $J(\omega_1) \leq J(\omega_2)$. Consequently, if we do not have the constraint on the volume of ω in (1.28), then $\inf_{\omega \in O_\beta} J(\omega) = J(\Omega)$.

Let us consider the following generalization of (1.27)

$$\begin{cases} -\Delta u_l(x) + l(x) u_l(x) = f(x) & \text{in } \Omega, \\ u_l(x) = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.29}$$

and

$$J(l) = \int_\Omega |\nabla u_l|^2 + l u_l^2 dx = \int_\Omega f u_l dx,$$

where $l \in L^\infty(\Omega)$. If we take $l \in \mathcal{R}_\beta$, then we will obtain the same problem as (1.27). We are going however to consider the relaxed problem, i.e., the minimization of $J(l)$ over

$$\bar{\mathcal{R}}_\beta = \left\{ l \in L^\infty(\Omega); 0 \leq l \leq 1, \int_\Omega l dx = \beta \right\}. \tag{1.30}$$

The existence theorem and characterization of minimizer are summarized as follows,

Theorem 1.6.1. *There exists $\hat{l}$ in $\bar{\mathcal{R}}_\beta$ which realizes the minimum of J in $\bar{\mathcal{R}}_\beta$. Also,*

$$\inf_{\omega \in O_\beta} J(\omega) = \min_{l \in \bar{\mathcal{R}}_\beta} J(l) = J(\hat{l}).$$

Moreover, let $l_0 \in \bar{\mathcal{R}}_\beta$ and $u_0 = u_{l_0}$. Then l_0 is a minimizer of J if and only if

1. *If $|\{0 < l < 1\}| > 0$, u_0 is constant on $\{0 < l < 1\}$.*

2. *For any $x_1 \in \{l_0 = 1\}$, $x_* \in \{0 < l < 1\}$ and $x_0 \in \{l = 0\}$, we have*

$$u_0(x_0) \leq u_0(x_*) \leq u_0(x_1).$$

The main difficulty is to find sufficient conditions on f and β, for the optimal design $\hat{l}$ to satisfy $\{0 < \hat{l} < 1\} = \emptyset$. This means that the relaxed problem with design set $\bar{\mathcal{R}}_\beta$ is identical with the classical problem with O_β. Thanks to the locality of classical Laplacian, [37] gives the following result in optimal domain where the uniqueness comes from Lemma 2.1.1.

Theorem 1.6.2. *Let $\tilde{u}$ denotes the solution of the auxiliary equations*

$$\begin{cases} -\Delta \tilde{u}(x) = f(x) & \text{in } \Omega, \\ \tilde{u}(x) = 0 & \text{on } \partial\Omega. \end{cases}$$

*There exists a characteristic function χ_E which is a minimum of the functional J if **one of** the following conditions holds.*

1. *$\tilde{u} \leq f$ in Ω.*

2. *$f \leq -\Delta f$ in Ω.*

3. *$\beta > |\{x \in \Omega; \tilde{u}(x) > \alpha\}|$, where $\alpha = \inf\{f(x); x$ such that $\tilde{u}(x) > f(x)\}$.*

Moreover, the minimizer χ_E is unique.

In Section 1.7 Theorem D, we have generalized the problem to the fractional case and proved the analogue of Theorem 1.6.1

1.7 Main Results

This work studies the maximization and minimization of a convex functional $\Phi_s(f)$ over $f \in \bar{\mathcal{R}}_\beta = \{0 \leq f \leq 1; \int_\Omega f dx = \beta\}$ which is the weak* closure of the set of rearrangements $\mathcal{R}_\beta$ in L^∞. The Gagliardo-Nirenberg seminorm $[\cdot]_s^2$

is defined in Definition 2.2.1. The functional $\Phi_s(f) := [u_f]_s^2$ is the Gagliardo-Nirenberg seminorm of u_f, which is the unique weak solution of the fractional Dirichlet boundary value problem with $0 < s < 1$ (see Chapter 2),

$$\begin{cases} (-\Delta)^s u_f(x) = f(x) \text{ in } \Omega, \\ u_f(x) = 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{1.31}$$

Our results ensure the existence of a maximizer on the one hand, and the existence and uniqueness of the minimizer on the other hand. The maximization problem is presented in Chapter 3 and is published in [8]; the minimization problem is presented in Chapter 4 and is published in [7].

Our main results are as follows:

Theorem A. *There exists a maximizer $\hat{f} \in \mathcal{R}_\beta$ such that*

$$\Phi_s(f) \leq \Phi_s(\hat{f})$$

for any $f \in \bar{\mathcal{R}}_\beta$. Moreover, for any maximizer $\hat{f} \in \bar{\mathcal{R}}_\beta$, there exists $\alpha > 0$ such that either $\hat{f} = \chi_{\{\hat{u}>\alpha\}}$ or $\hat{f} = \chi_{\{\hat{u}\geq\alpha\}}$, where $\hat{u} = u_{\hat{f}}$.

The formulation of Theorem A slightly differs from the Theorem 1.2 in [8]. This is due to a minor mistake recently discovered. In Chapter 3 we present the corrected proof, and a corrigendum paper is published in [21].

Theorem B. *There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$ such that*

$$\Phi_s(\hat{f}) \leq \Phi_s(f)$$

for any $f \in \bar{\mathcal{R}}_\beta$. Moreover, $\hat{f} > 0$ a.e. in Ω, and for some $\alpha > 0$, the function $\hat{u} = u_{\hat{f}}$ satisfies

$$0 \leq \hat{u} \leq \alpha,$$
$$\{\hat{f} < 1\} \subset \{\hat{u} = \alpha\},$$
$$\{\hat{u} < \alpha\} \subset \{\hat{f} = 1\}.$$

Also, the function $\hat{U} = \alpha - \hat{u}$ minimizes the convex functional

$$J(v) = [v]_s^2 + \int_\Omega v^+ dx,$$

among all v such that $\alpha - v \in H_0^s(\Omega)$, and $\hat{U}$ satisfies the following inequalities

$$\chi_{\{U>0\}} \leq -(-\Delta)^s U \leq \chi_{\{U\geq 0\}}.$$

Additionally, $\hat{U}$ is the solution to the following equation

$$-(-\Delta)^s U - \chi_{\{U\leq 0\}} \min\left\{-(-\Delta)^s U^+ ; 1\right\} = \chi_{\{U>0\}} \quad \text{in } \Omega.$$

Moreover, we bring a constant in fractional operators and the Gagliardo-Nirenberg seminorm (see Section 4.4.1). Also, let the sequence $\{s_k\} \in (0,1)$ be such that $s_k \to 1^-$ as $k \to \infty$. In this case, $u_k := u_{f_k}$ is the solution to the modified fractional problem

$$\begin{cases} (-\Delta)^{s_k} u_k(x) = f_k(x) \text{ in } \Omega, \\ u_k(x) = 0 \qquad\qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $f_k \in \bar{\mathcal{R}}_{\tilde{\beta}}$ (see Section 4.4.2). The functional of the minimization problem is denoted by $\Phi_{s_k} := [u_k]^2_{s_k}$, where $[\cdot]^2_{s_k}$ is the modified Gagliardo-Nirenberg seminorm (see (4.26)). The optimal load of Φ_{s_k} is denoted by $\hat{f}_k$, and $\hat{u}_k = u_{\hat{f}_k}$.

What is more, let $f \in \bar{\mathcal{R}}_\beta$ and

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx,$$

where u_f is the solution to

$$\begin{cases} -\Delta u_f = f \quad \text{in } \Omega, \\ u = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Also, the optimal load of Φ is denoted by f^*, and $u^* = u_{f^*}$. Our main result is as follows,

Theorem C. *As $k \to \infty$, we obtain the following up to a subsequence,*

1. $\hat{f}_k \overset{*}{\rightharpoonup} C(n) f^*$ *in* $L^\infty(\Omega)$,
2. $\Phi_{s_k}(\hat{f}_k) \to C(n)\Phi(f^*)$,
3. $\hat{u}_k \to u^*$ *in* $L^2(\Omega)$,

where $C(n)$ is a constant depending only on dimension.

These results are proven in Chapter 3 and Chapter 4, where Theorem A occurs as Theorem 3.1.1, Theorem B occurs with slightly more details as Theorem 4.2.1, Theorem 4.3.1, and Theorem 4.3.2 and Theorem C occurs as Theorem 4.4.1.

In addition, we consider the variational Dirichlet boundary value fractional Laplacian equation with $0 < s < 1$

$$\begin{cases} (-\Delta)^s u_l(x) + l u_l(x) = f(x) \qquad \text{in } \Omega, \\ u_l(x) = 0 \qquad\qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{1.32}$$

where $l \in \bar{\mathcal{R}}_\beta$ and f is a positive given function. We study the minimization of the convex functional

$$J(l) = [u_l]^2_s + \int_\Omega l u_l^2 dx,$$

where $[\cdot]_s$ is the Gagliardo-Nirenberg seminorm.

Theorem D. *There exists a minimizer* $\hat{l} \in \bar{\mathcal{R}}_\beta$ *such that*

$$J(\hat{l}) \leq J(l)$$

for any $l \in \bar{\mathcal{R}}_\beta$*. Moreover,* $\hat{u} = u_{\hat{l}}$ *is constant in* $\{0 < \hat{l} < 1\}$*, and for any*

$$x_1 \in \{\hat{l} = 1\}, x_* \in \{0 < \hat{l} < 1\}, x_0 \in \{\hat{l} = 0\},$$

we have $\hat{u}(x_0) \leq \hat{u}(x_*) \leq \hat{u}(x_1)$*.*

This result is proven in Chapter 5, where Theorem D occurs as Theorem 5.3.1 and Theorem 5.3.2.

Chapter 2

Preliminaries

In this Chapter, we will introduce some preliminaries in analysis, fractional calculus and $\Gamma-$convergence.

2.1 Definitions and Properties from Analysis

Definition 2.1.1. *An extreme point of a convex set A in a real vector space is a point in A which does not lie in any line segment joining two points of A, i.e., $a \in ext(A)$, if for any $t \in (0,1)$ and $a_1, a_2 \in A$, $a = ta_1 + (t-1)a_2$ implies $a_1 = a_2 = a$.*

Recall that

$$\mathcal{R}_\beta = \{f : f = \chi_E, |E| = \beta\},$$

and

$$\bar{\mathcal{R}}_\beta = \left\{ f : 0 \leq f \leq 1, \int_\Omega f dx = \beta \right\} \subset L^\infty(\Omega),$$

which is the weak* closed convex hull of $\mathcal{R}_\beta$. The following Lemma implies that the collection of *extreme points* of $\bar{\mathcal{R}}_\beta$ is exactly $\mathcal{R}_\beta$.

Lemma 2.1.1. *$f \in \bar{\mathcal{R}}_\beta$ is a extreme point of $\bar{\mathcal{R}}_\beta$ if and only if f is characteristic function, or $ext(\bar{\mathcal{R}}_\beta) = \mathcal{R}_\beta$.*

This is a well-known results, but unfortunately we cannot find the original reference. For sake of completeness, we present the proof.

Proof. Assume $f = \chi_E$ with $\mathcal{L}^n(E) = \beta$ and $f = \alpha g_1 + (1-\alpha) g_2$ where $g_1, g_2 \in \bar{\mathcal{R}}_\beta$ and $\alpha \in (0,1)$. If $x \notin E$, $g_1(x) = g_2(x) = 0$. Hence, $g_1 = g_2 = f$, and f is an extreme point of $\bar{\mathcal{R}}_\beta$.

On the other hand, assume f is an extreme point of $\bar{\mathcal{R}}_\beta$ and $f \notin \mathcal{R}_\beta$. Then there exists $\delta > 0$ such that $\mathcal{L}^n(A_\delta) > 0$ with $A_\delta = \{x \mid \delta < f < 1-\delta\}$. We may decompose A_δ with the following method,

$$A_\delta = A_\delta^+ \cup A_\delta^- \ ; \ A_\delta^+ \cap A_\delta^- = \emptyset \ ; \ \mathcal{L}^n(A_\delta^+) = \mathcal{L}^n(A_\delta^-) = \frac{1}{2}\mathcal{L}^n(A_\delta).$$

Choose $\varepsilon \in (0, \delta)$ and define

$$h_\varepsilon^\pm(x) = \begin{cases} f(x) \ x \notin A_\delta \\ f(x) \pm \varepsilon \ x \in A_\delta^+ \\ f(x) \mp \varepsilon \ x \in A_\delta^- \end{cases}$$

Direct Observation gives $h_\varepsilon^\pm(x) \in \bar{\mathcal{R}}_\beta$ and $f = 1/2h_\varepsilon^+ + 1/2h_\varepsilon^-$, which is a contradiction. □

According to [16, Lemma 2.4], we formulate the following Lemma.

Lemma 2.1.2. *We have*

$$L_+^2(\Omega) = \left\{g \in L^2(\Omega); g \geq 0 \ a.e.\right\}.$$

If $g \in L_+^2(\Omega)$, then there exists $f \in ext(\bar{\mathcal{R}}_\beta) = \mathcal{R}_\beta$ such that

$$\int_\Omega hg \leq \int_\Omega fg,$$

for all $h \in \bar{\mathcal{R}}_\beta$.

The following lemma from [41] is applied significantly in rearrangement problems.

Lemma 2.1.3 (Bathtub Lemma)**.** *Let (Ω, σ, μ) be a sigma-finite measure space and let h be a real-valued, measurable function on Ω such that*

$$\mu\left(\{x : h(x) < t\}\right) \quad \text{is finite for all } t \in \mathbb{R}.$$

Let the number $\beta > 0$ be given and define a class of measurable functions on Ω by

$$\bar{\mathcal{R}}_\beta = \left\{f : 0 \leq f(x) \leq 1 \text{ and } \int_\Omega f(x)\mu(dx) = \beta\right\}.$$

Then the minimization problem

$$I = \inf_{f \in \bar{\mathcal{R}}_\beta} \int_\Omega h(x)f(x)\mu(dx)$$

is solved by

$$f(x) = \chi_{\{h<s\}}(x) + c\chi_{\{h=s\}}(x),$$

where

$$s = \sup\left\{t : \mu(\{x : h(x) < t\}) \leq \beta\right\},$$

$$c\mu(\{x : h(x) = s\}) = \beta - \mu(\{x : h(x) < s\}).$$

2.2 Fractional Laplacian

In this section, we provide preliminaries of *fractional Sobelov Space* and *fractional operators* $(-\Delta)^s$ which are used throughout this work.

Now, we introduce *the fractional Sobolev space.*

Definition 2.2.1. *For $0 < s < 1$, we define the fractional Sobolev space as following:*

$$H^s(\mathbb{R}^n) = \left\{ v \in L^2(\mathbb{R}) : [v]_s^2 < \infty \right\},$$

where $[\cdot]_s$ is the Gagliardo-Nirenberg semi-norm,

$$[v]_s^2 = \iint_{\mathbb{R}^{n\times n}} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}} dxdy.$$

Also, we denote by $H^{-s}(\mathbb{R})$ the dual space to $H^s(\mathbb{R}^n)$. In other words, f belongs to $H^{-s}(\mathbb{R}^n)$ provided f is a bounded linear functional on $H^s(\mathbb{R}^n)$. Moreover, for a open bounded domain $\Omega \subset \mathbb{R}^n$,

$$H_0^s(\Omega) = \{v \in H^s(\mathbb{R}^n) : v(x) = 0 \quad in \ \ \mathbb{R}^n \setminus \Omega\},$$

and $H^{-s}(\Omega)$ is the dual space of $H_0^s(\Omega)$. Then we have

$$H_0^s(\Omega) \subset H^s(\mathbb{R}^n) \subset H^{-s}(\mathbb{R}^n) \subset H^{-s}(\Omega).$$

Proposition 2.2.1. *The Gagliardo-Nirenberg semi-norm is Gâteaux-differentiable.*

Proof. Let us first check that $[v]_s$ is a semi-norm. Clearly, for any $\lambda \in \mathbb{R}$,

$$[\lambda v]_s = |\lambda|[v]_s.$$

Next assume $u, v \in H^s(\mathbb{R}^n)$. Then, note that,

$$\begin{aligned} [u+v]_s^2 &= \iint_{\mathbb{R}^{n\times n}} \frac{|u(x) + v(x) - u(y) - v(y)|^2}{|x-y|^{n+2s}} dxdy \\ &= [u]_s^2 + [v]_s^2 \\ &+ 2\iint_{\mathbb{R}^{n\times n}} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x-y|^{n+2s}} dxdy. \end{aligned}$$

Hence, it suffices to prove

$$\int_{\mathbb{R}^{n\times n}} \frac{(u(x) - u(y)) \cdot (v(x) - v(y))}{|x-y|^{n+2s}} dxdx \leq [u]_s \cdot [v]_s.$$

Indeed, this follows directly from *Cauchy-Schwartz inequality.*

Secondly, we claim that $H^s(\mathbb{R}^n)$ is a Banach space with the norm,

$$\|u\|_s = \left(\|u\|_2^2 + [u]_s^2\right)^{1/2}.$$

The proof is classical and is omitted here (see [30]).

Therefore, the Gâteaux derivative of $[\cdot]_s^2$ at u is,

$$\begin{aligned} & \lim_{\varepsilon\to 0}\varepsilon^{-1}\left([u+\varepsilon v]_s^2-[u]_s^2\right) \\ = \;& 2\iint_{\mathbb{R}^{n\times n}}\frac{(u(x)-u(y))\cdot(v(x)-v(y))}{|x-y|^{n+2s}}dxdy. \end{aligned} \tag{2.1}$$

□

The following is the so-called singular integral definition of fractional Laplace operators (see [30]). The abbreviation "p.v." stands for "principle value".

Definition 2.2.2. *For a function $u \in H^s(\mathbb{R}^n)$ and any $x \in \mathbb{R}^n$, the fractional Laplace operators is,*

$$(-\Delta)^s u(x) = p.v.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy = \lim_{\varepsilon\to 0}(-\Delta)^s_\varepsilon u(x), \tag{2.2}$$

where

$$(-\Delta)^s_\varepsilon u(x) = \int_{\mathbb{R}^n\setminus B_\varepsilon(x)}\frac{u(x)-u(y)}{|x-y|^{n+2s}}dy.$$

Definition 2.2.3. *A function u is $s-$harmonic in Ω if $(-\Delta)^s u = 0$ for any $x \in \Omega$.*

Let $\mathcal{S}$ be the collection of Schwartz functions, which is referred in [12, Section 3.1].

Definition 2.2.4. *The Schwartz space on $\mathbb{R}^n$ is the function space*

$$\mathcal{S} = \left\{ f \in C^\infty(\mathbb{R}^n); \sup_{x\in\mathbb{R}^n}\left|x^\alpha\left(D^\beta f\right)(x)\right| < \infty\right\},$$

where α and β are arbitrary multi-indices with n entries.

By [30], the following Lemma gives an alternative definition of the fractional Laplace operators for $u \in \mathcal{S}$ and we reproduce the proof here.

Lemma 2.2.1. *Let $(-\Delta)^s$ be the fractional Laplace operators defined in Definition 2.2.2. Then for any $u \in \mathcal{S}$, we have*

$$(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^n}\frac{2u(x)-u(x+y)-u(x-y)}{|y|^{n+2s}}dy, \tag{2.3}$$

for any $x \in \mathbb{R}^n$.

Proof. We begin by change of variables in (2.2). First we take $z = y - x$ and then by symmetry, take $t = -z$. We have

$$\begin{aligned}
& \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy \\
= & \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz \\
= & \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(x-t)}{|t|^{n+2s}} dt
\end{aligned}$$

Now, we sum them up and relabel z and t. We obtain

$$\begin{aligned}
& 2(-\Delta)^s u(x) \\
= & \text{ p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(x+z)}{|z|^{n+2s}} dz + \text{p.v.} \int_{\mathbb{R}^n} \frac{u(x) - u(x-t)}{|t|^{n+2s}} dt \\
= & \text{ p.v.} \int_{\mathbb{R}^n} \frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} dy. \qquad (2.4)
\end{aligned}$$

It suffices to show that the integral is well-posed in $\mathbb{R}^n$. Indeed, for any $u \in C^\infty$, Taylor expansion gives,

$$u(x+y) + u(x-y) = 2u(x) + y^\top \left\{ D^2 u(x) \right\} y + o(|y|^2),$$

where $D^2 u(x)$ is the Hessian matrix. Now, by taking the limit $|y| \to 0$, we obtain

$$\frac{2u(x) - u(x+y) - u(x-y)}{|y|^{n+2s}} \leq \frac{\|D^2 u(x)\|_\infty}{|y|^{n+2s-2}},$$

where $\|D^2 u(x)\|_\infty$ is the maximum absolute row sum of the Hessian matrix. Thus (2.4) is integrable near 0. Hence, we can remove the principle value in (2.4) and the proof is complete. ☐

In [6, Definition 1.1], we have the following definition of *weak solution*.

Definition 2.2.5. *For a function $f \in H^{-s}(\Omega)$, we say that $u_f \in H_0^s(\Omega)$ is a weak solution of the fractional boundary value problem with homogeneous Dirichlet boundary condition*

$$\begin{cases} (-\Delta)^s u_f(x) = f(x) & \text{in } \Omega, \\ u_f(x) = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \qquad (2.5)$$

if

$$\iint_{\mathbb{R}^{n \times n}} \frac{(u_f(x) - u_f(y)) \cdot (v(x) - v(y))}{|x-y|^{n+2s}} dx dy = \int_\Omega f(x) v(x) dx, \qquad (2.6)$$

for any $v \in H_0^s(\Omega)$.

Given $f \in H^{-s}(\Omega)$, let u_f be the weak solution of (2.5) and let us define the functional

$$\Phi_s(f) = [u_f]_s^2. \tag{2.7}$$

Also, note that for any $\varphi \in H_0^s(\Omega)$,

$$\begin{aligned} \langle(-\Delta)_\varepsilon^s u, \varphi\rangle &= \int_{\mathbb{R}^n} (-\Delta)_\varepsilon^s u(x)\varphi(x)dx \\ &= \int_{\mathbb{R}^n}\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\varphi(x)dxdy \\ &= \int_{\mathbb{R}^n}\int_{|x-y|>\varepsilon} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\left(-\varphi(x)\right)dxdy \\ &= \frac{1}{2}\int_{\mathbb{R}^n}\int_{|x-y|>\varepsilon} \frac{(u(x)-u(x))\cdot(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy. \end{aligned}$$

Lemma 2.2.2. *For any $f \in H^{-s}(\Omega)$, (2.5) has a unique weak solution u_f, which satisfies*

$$\int_\Omega f u_f dx = [u_f]_s^2 = \sup_{u \in H_0^s(\Omega)} \left\{ 2\int_\Omega f u dx - [u]_s^2 \right\}. \tag{2.8}$$

Proof. If u_f is a solution of (2.5), the first equation of (2.8) follows from (2.6). For any $u \in H_0^s(\Omega)$, we take

$$\Psi(u) := [u]_s^2 - 2\int_\Omega f u dx.$$

Since Ψ is strictly convex, there exists a unique minimizer of Ψ, say $u_0 \in H_0^s(\Omega)$. It suffices to show that u_0 is a weak solution of (2.5) if and only if u_0 minimizes Ψ. Take $u_\varepsilon := u_0 + \varepsilon\varphi$ for $\varepsilon \in \mathbb{R}, \varphi \in H_0^s(\Omega)$. Then, $\Psi(u_\varepsilon) \geq \Psi(u_0)$ implies

$$\begin{aligned} &\iint_{\mathbb{R}^{n\times n}} \frac{2\varepsilon\left(u_0(x)\varphi(x) + u(y)\varphi(y) - u(x)\varphi(y) - u(y)\varphi(x)\right)}{|x-y|^{n+2s}}dxdy \\ - &\ 2\varepsilon\int_\Omega f\varphi dx + o(\varepsilon) \\ = &\ 2\varepsilon\left(\langle(-\Delta)^s u_0, \varphi\rangle - \int_\Omega f\varphi dx\right) + o(\varepsilon) \\ \geq &\ 0, \quad \text{for any } \varepsilon \in \mathbb{R}. \end{aligned}$$

From above, we obtain

$$\langle(-\Delta)^s u_0, \varphi\rangle = \int_\Omega f\varphi dx,$$

as required.

Conversely, assume $u_0 = u_f$ is the solution of (2.5). Then for an arbitrary $u_1 \in H_0^s(\Omega)$, a direct computation gives

$$\begin{aligned}
& \Psi(u_1) - \Psi(u_0) \\
= \;& [u_1]_s^2 + [u_f]_s^2 - 2\int_\Omega f u_1 dx \\
= \;& \iint_{\mathbb{R}^{n\times n}} \frac{(u_1(x) - u_f(x) - u_1(y) + u_f(y))^2}{|x-y|^{n+2s}} dxdy \\
\geq \;& 0.
\end{aligned}$$

The proof is complete. □

We refer to [10, Lemma 2.4] for the Poincaré-type inequality for the fractional Gagliardo-Nirenberg seminorm.

Lemma 2.2.3. *Let $s \in (0,1)$, $\Omega \in \mathbb{R}^n$ be an open and bounded set. Then we have,*

$$\|u\|_{L^2(\Omega)}^2 \leq C(n,s,\Omega)[u]_s^2, \quad \textit{for every } \; u \in H_0^s(\Omega),$$

where the geometric quantity $C(n,s,\Omega)$ is defined by

$$C(n,s,\Omega) = \min\left\{ \frac{diam\,(\Omega \cup B)^{n+2s}}{|B|} : B \subset \mathbb{R}^n \backslash \Omega \;\; \textit{is a ball} \right\}. \tag{2.9}$$

With analogue to *Rellich-Kondrachov theorem*, [46] gives the following *Fractional Compact Embedding Theorem*,

Lemma 2.2.4. *Let $n \geq 1$, $\Omega \in \mathbb{R}^n$ be a Lipschitz open bounded set and $\mathfrak{J}$ be a bounded subset of $L^2(\Omega)$. Suppose that*

$$\sup_{f\in\mathfrak{J}} \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^2}{|x-y|^{n+2s}} dxdy < \infty.$$

Then, $\mathfrak{J}$ is pre-compact in $L^2(\Omega)$.

For Ω bounded, $s \in (0,1)$ and $g \in L^\infty(\Omega)$, consider a more general Dicichlet problem comparing with (2.5),

$$\begin{cases} (-\Delta)^s u_g(x) = g(x) & \text{in } \Omega, \\ u_g(x) = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{2.10}$$

The global regularity of the weak solution of (2.10) has been discussed in [50].

Proposition 2.2.2. *Let $w = (-\Delta)^s u$. Assume $w \in L^\infty(\mathbb{R}^n)$ and $u \in L^\infty(\mathbb{R}^n)$. Then,*

1. *If $2s < 1$, then $u \in C^{0,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s$. Moreover,*

$$\|u\|_{C^{0,\alpha}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^\infty} + \|w\|_{L^\infty}\right),$$

   where C depends only on n, α and s.

2. *If $2s > 1$, then $u \in C^{1,\alpha}(\mathbb{R}^n)$ for any $\alpha < 2s - 1$. Moreover,*

$$\|u\|_{C^{1,\alpha}(\mathbb{R}^n)} \leq C\left(\|u\|_{L^\infty} + \|w\|_{L^\infty}\right),$$

   where C depends only on n, α and s.

It has been observed in [46] that the above results are valid also for solution of $(-\Delta)^s u = f$ in bounded domains. Up to the boundary C^s-regularity has been proven in [48], which implies that the weak solution of (2.5) is continuous in $\mathbb{R}^n$. We say that Ω satisfies uniform exterior sphere condition if there exists $r > 0$ such that for any $x \in \partial\Omega$ there exists $y \in \mathbb{R}^n$ for which $B(y,r) \cap \Omega = \emptyset$ and $x \in \partial B(y,r)$. By [3], any domain with $C^{1,1}$ boundary in $\mathbb{R}^n$ satisfies uniform exterior sphere condition. The following can be found in [48, Proposition 1.1].

Proposition 2.2.3. *Let Ω be a bounded Lipschitz domain with uniform exterior sphere condition, $g \in L^\infty(\Omega)$, and u be the weak solution of* (2.10). *Then $u \in C^s(\mathbb{R}^n)$ and*

$$\|u\|_{C^s} \leq C\|g\|_{L^\infty(\Omega)},$$

where C is a constant depending on s and Ω.

2.3 $\Gamma-$convergence Toolbox

In this section, we introduce the definition of $\Gamma-$convergence and the so-called *coercive condition.* These concepts were introduced by De Giorgi in the 60s and is now a well-understood tool to deal with the convergence of minimum problems. Here we cite [9] and [29].

Definition 2.3.1. *Let X be a metric space. We say that a sequence $f_j : X \to \bar{\mathbb{R}}$ $\Gamma-$converges in X to $f_\infty : X \to \bar{\mathbb{R}}$ if for all $x \in X$ we have*

1. *(lim inf inequality) for every sequence $\{x_j\}$ converging to x*

$$f_\infty(x) \leq \liminf_{j\to\infty} f_j\{x_j\};$$

2. *(lim sup inequality) there exists a sequence $\{x_j\}$ converging to x such that*

$$f_\infty \geq \limsup_{j\to\infty} f_j\{x_j\}.$$

The function f_∞ is called the $\Gamma-$limit of $\{f_j\}$, and we write

$$f_\infty = \Gamma - \lim_{j\to\infty} f_j.$$

It is well-known that the $\Gamma-$limit is unique if it exists.

Definition 2.3.2. *We say a sequence $f_j : X \to \bar{\mathbb{R}}$ is equi-mildly coercive if there exists a non-empty compact set $K \subset X$ such that $\inf_X f_j = \inf_K f_j$ for all j.*

Finally, we have the following result.

Theorem 2.3.1. *Let (X, d) be a metric space, let $\{f_j\}$ be a sequence of equi-mildly coercive functions on X, and let $f_\infty = \Gamma - \lim_{j\to\infty} f_j$. Then the minimum point of f_∞ exists and*

$$\min_X f_\infty = \lim_{j\to\infty} \inf_X f_j.$$

Moreover, if $\{x_j\}$ is a precompact sequence such that

$$\lim_{j\to\infty} f_j(x_j) = \lim_{j\to\infty} \inf_X f_j,$$

then every limit of a subsequence of $\{x_j\}$ is a minimum point of f_∞.

Chapter 3

The Maximization Problem

In this chapter, we consider the fractional analogue of the optimal rearrangement problem and show that its maximizers solve the fractional unstable obstacle problem that has been recently considered in [2].

3.1 Problem Setting

The setting of this fractional problem is the following. Let $0 < s < 1$ be fixed and Ω be a bounded, open domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. To avoid extra notations, in this chapter we will use u_f to denote the solution to

$$\begin{cases} (-\Delta)^s u_f(x) = f(x) & \text{in } \Omega, \\ u_f = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $f \in \bar{\mathcal{R}}_\beta$. Also, we define the functional

$$\Phi_s(f) := [u_f]_s^2,$$

where $[\cdot]$ is the Gagliardo-Nirenberg seminorm (see Definition 2.2.1).

The main result is the following.

Theorem 3.1.1. *There exists a maximizer $\hat{f} \in \mathcal{R}_\beta$ such that*

$$\Phi_s(f) \leq \Phi_s(\hat{f})$$

for any $f \in \bar{\mathcal{R}}_\beta$. Moreover, for any maximizer $\hat{f} \in \bar{\mathcal{R}}_\beta$ of Φ_s there exists $\alpha > 0$ such that

$$\text{either } \hat{f} = \chi_{\{\hat{u} > \alpha\}} \text{ or } \hat{f} = \chi_{\{\hat{u} \geq \alpha\}}$$

where $\hat{u} = u_{\hat{f}}$.

As a result the function $\hat{u}$ solves one of the following fractional unstable obstacle equations:

$$\begin{aligned} &\text{either} && (-\Delta)^s \hat{u} = \chi_{\{\hat{u} > \alpha\}}, \\ &\text{or} && (-\Delta)^s \hat{u} = \chi_{\{\hat{u} \geq \alpha\}}. \end{aligned} \tag{3.1}$$

3.2 Proof of Theorem 3.1.1

Our method is based on classical approach, but due to non-local character of the problem, new techniques need to be developed in proving (3.1). We will divide the proof into a series of lemmas and we start with the following existence result.

Lemma 3.2.1. *There exists a maximizer $\hat{f} \in \mathcal{R}_\beta$.*

Proof. Let

$$I := \sup_{f \in \bar{\mathcal{R}}_\beta} \int_\Omega f u_f dx.$$

We first show that I is finite. Consider $f \in \bar{\mathcal{R}}_\beta$. Then, by Lemma 2.2.2, u_f satisfies

$$\int_\Omega f u_f dx = [u_f]_s^2. \tag{3.2}$$

Using *Cauchy-Schwartz inequality* and Lemma 2.2.3,

$$\int_\Omega f u_f dx \leq \|f\|_2 C [u_f]_s, \tag{3.3}$$

and thus we obtain the following with the fact that $f \in [0, 1]$ in Ω

$$\int_\Omega f u_f dx \leq C \|f\|_2^2 \leq C, \tag{3.4}$$

which proves that I is finite.

Let now $\{f_i\}_{i \in \mathbb{N}} \subset \bar{\mathcal{R}}_\beta$ be a maximization sequence and let $u_i := u_{f_i}$. Then,

$$I = \lim_{i \to \infty} \int_\Omega f_i u_i dx.$$

It is clear from (3.2) and (3.4) that $\{u_i\}$ is bounded in $H_0^s(\Omega)$. Hence by Lemma 2.2.4, there exists a subsequence (still denoted by $\{u_i\}$) and some $u_0 \in L^2(\Omega)$ such that $\{u_i\}$ converges strongly to u_0 and weakly in $H_0^s(\Omega)$. Since $[\cdot]_s^2$ is convex, it follows that it is sequentially weakly lower semi-continuous and hence $u_0 \in H_0^s(\Omega)$ and,

$$[u_0]_s^2 \leq \liminf_{i \to \infty} \, [u_i]_s^2 = I. \tag{3.5}$$

On the other hand, since $\{f_i\}$ is both bounded in $L^2(\Omega)$ and $L^\infty(\Omega)$, there exists a subsequence (still denoted by $\{f_i\}$) converging weakly in $L^2(\Omega)$ and weakly* in $L^\infty(\Omega)$ to some $\eta \in L^\infty(\Omega)$. Since $\bar{\mathcal{R}}_\beta$ is weakly* closed, we have $\eta \in \bar{\mathcal{R}}_\beta$. Thus, we obtain

$$\int_\Omega f_i u_i \, dx \to \int_\Omega \eta u_0 \, dx. \tag{3.6}$$

By Lemma 2.2.2, (3.5) and (3.6), we obtain

$$\begin{aligned} I &= \lim_{i\to\infty} \int_D f_i u_i dx = \lim_{i\to\infty} 2\int_D u_i f_i \, dx - [u_i]_s^2 \\ &\leq 2\int_D u_0 \eta \, dx - [u_0]_s^2. \end{aligned} \tag{3.7}$$

According to Lemma 2.1.2, there exists $\hat{f} \in \mathcal{R}_\beta$ such that

$$\int_\Omega \hat{f} u_0 dx = \sup_{h\in\bar{\mathcal{R}}_\beta} \int_\Omega h u_0 dx. \tag{3.8}$$

Applying again Lemma 2.2.2 together with (3.7) and (3.8), we obtain

$$\begin{aligned} I &\leq 2\int_\Omega \hat{f} u_0 \, dx - [u_0]_s^2 \\ &\leq 2\int_\Omega \hat{f}\hat{u} - [\hat{u}]_s^2 \\ &= \int_\Omega \hat{f}\hat{u} dx \leq I, \end{aligned}$$

where $\hat{u} = u_{\hat{f}}$. Thus, $\hat{f}$ is a minimizer of Φ_s. □

From now on, $\hat{f}$ will denote **any** minimizer of Φ_s, not necessary the one obtained in Lemma 3.2.1, which we already know belonging to $\mathcal{R}_\beta$.

Lemma 3.2.2. *$\hat{f}$ maximizes the linear functional $L(f) := \int_\Omega \hat{u} f dx$ over $\bar{\mathcal{R}}_\beta$, where $\hat{u} = u_{\hat{f}}$.*

Proof. Let us take any $f \in \bar{\mathcal{R}}_\beta$ and let u_f be the solution. We use the maximization property

$$\Phi_s\left((1-\varepsilon)\hat{f} + \varepsilon f\right) \leq \Phi_s(\hat{f}),$$

where $\varepsilon \in [0,1]$. This inequality implies that

$$\begin{aligned} & 2\varepsilon \iint_{\mathbb{R}^{n\times n}} \frac{(\hat{u}(x)-\hat{u}(y))\left((u_f-\hat{u})(x)-(u_f-\hat{u})(y)\right)}{|x-y|^{n+2s}} dxdy \\ + \ & \varepsilon^2 [u_f - \hat{u}]_s^2 \leq 0. \end{aligned}$$

If we divide by ε and take the limit $\varepsilon \to 0^+$, we get

$$\iint_{\mathbb{R}^{n\times n}} \frac{(\hat{u}(x)-\hat{u}(y))\,(u(x)-u(y))}{|x-y|^{n+2s}} dxdy \leq [\hat{u}]_s^2.$$

Now if we use Lemma 2.2.2, the last inequality becomes

$$\int_\Omega f\hat{u} dx \leq \int_\Omega \hat{f}\hat{u} dx,$$

as we wanted to show. □

Next, observe that from Lemma 2.1.2 there exists a $\tilde{f} = \chi_E \in \mathcal{R}_\beta = \text{ext}(\bar{\mathcal{R}}_\beta)$ such that $\tilde{f}$ maximizes $L(f)$ over $\bar{\mathcal{R}}_\beta$.

Lemma 3.2.3. *Let* $\alpha := \sup_{x\in\mathbb{R}^n\setminus E} \hat{u}(x)$ *and* $\gamma := \inf_{x\in E} \hat{u}(x)$ *(where* sup *and* inf *denote the essential supremum and the essential infimum respectively). Then,* $\alpha \le \gamma$.

Proof. Assume by contradiction that $\gamma < \alpha$. Let us fix $\gamma < \xi_1 < \xi_2 < \alpha$. Since $\xi_1 > \gamma$, there exists a set $A \subset E$, with positive measure, such that $\hat{u} \le \xi_1$ on A. Similarly, $\xi_2 < \alpha$ implies that there exists a $B \subset \mathbb{R}^n \setminus E$, with positive measure, such that $\hat{u} \ge \xi_2$ on B. Without loss of generality, we assume that A and B have the same Lebesgue measure. Next, we define a new rearrangement of $\tilde{f}$, which is denoted by $\bar{f}$,

$$\bar{f} = \begin{cases} 0, & x \in A; \\ 1, & x \in B; \\ \tilde{f}(x), & x \in \Omega \setminus (A \cup B)\,. \end{cases}$$

Therefore,

$$\begin{aligned} & \int_\Omega \bar{f}\hat{u}dx - \int_\Omega \hat{f}\hat{u}dx \\ = \;& \int_B \bar{f}\hat{u}dx - \int_A \hat{f}\hat{u}dx \\ \ge \;& \xi_2 \int_B \bar{f}dx - \xi_1 \int_A \tilde{f}dx \\ = \;& (\xi_2 - \xi_1) \int_A \tilde{f}dx > 0, \end{aligned}$$

which contradicts the maximality of $\hat{f}$. □

Since $\hat{u}$ is continuous in $\mathbb{R}^n$ (see Section 2.2), we have $\alpha = \gamma$ in Lemma 3.2.3.

Lemma 3.2.4. $\chi_{\{\hat{u}>\alpha\}} \le \hat{f} \le \chi_{\{\hat{u}\ge\alpha\}}$.

Proof. It suffices to prove that

$$\hat{f} = \begin{cases} 1 & a.e. \text{ in } \{\hat{u} > \alpha\}\,, \\ 0 & a.e. \text{ in } \{\hat{u} < \alpha\}\,. \end{cases}$$

We argue by contradiction. Assume there exists a $A \subset \{\hat{u} > \alpha\}$, with positive measure, such that $\hat{f} < 1$ in A. Since $|\{\hat{u} > \alpha\}| \le \beta$, $\hat{f} > 0$ in some subset of $\{\hat{u} \le \alpha\}$. Thus, we can replace the function $\hat{f}$ by a function $f \in \bar{\mathcal{R}}_\beta$ which has larger values in A and smaller values in $\{\hat{u} \le \alpha\}$. As a result,

$$\int_\Omega f\hat{u}dx > \int_\Omega \hat{f}\hat{u}dx,$$

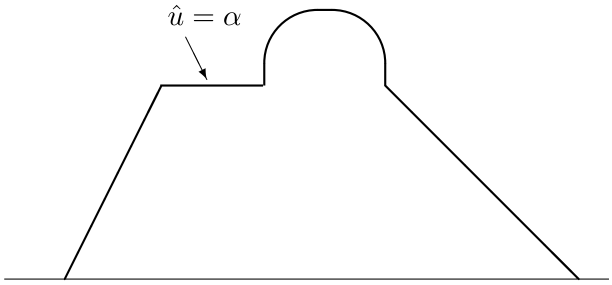


Figure 3.1: The function $\hat{u}$

which contradicts the maximality of $\hat{f}$. Therefore, $\hat{f} = 1$ a.e. in $\{\hat{u} > \alpha\}$.

Similarly, assume there exists a $A \subset \{\hat{u} < \alpha\}$, with positive measure, such that $\hat{f} > 0$ in A. Since $E \subset \{\hat{u} \geq \alpha\}$, $\hat{f} < 1$ in some subset of $\{\hat{u} \geq \alpha\}$. Thus, we can replace the function $\hat{f}$ by a function $f \in \bar{\mathcal{R}}_\beta$ which vanishes in A and has larger values in $\{\hat{u} \geq \alpha\}$. As a result,

$$\int_\Omega f\hat{u}dx > \int_\Omega \hat{f}\hat{u}dx,$$

which contradicts the maximality of $\hat{f}$. Therefore, $\hat{f} = 0$ a.e. in $\{\hat{u} < \alpha\}$. □

Now, we are in the position to study the shape optimization problems. A natural question arises: can we show that the maximizer $\hat{f}$ is actually a characteristic function. It is equivalent to ask whether

$$\left|\{\hat{f} \in (0,1)\}\right| = 0.$$

Thanks to Lemma 3.2.4, it suffices to prove $f = 1$ in $\{\hat{u} = \alpha\}$. In fractional settings, the difficulty arises from the non-locality. So the following lemma seems to be rather original.

Lemma 3.2.5. *Either* $\hat{f} = \chi_{\{\hat{u}>\alpha\}}$ *or* $\hat{f} = \chi_{\{\hat{u}\geq\alpha\}}$.

Proof. If $|\{u = \alpha\}| = 0$ then Claim 4 implies $f = \chi_{\{u>\alpha\}}$ a.e.. If $|\{u = \alpha\}| > 0$ and $|\{u < \alpha\}| = |\Omega| - \beta$, then Claim 4 implies $f = \chi_{\{u\geq\alpha\}}$ a.e.. If $|\{u = \alpha\}| > 0$ and $|\{u < \alpha\}| < |\Omega| - \beta$, then it is easy to show that there exists a subset $\tilde{E} \subset \Omega$ such that $\{\hat{u} > \alpha\} \subsetneqq \tilde{E} \subsetneqq \{\hat{u} \geq \alpha\}$, $|\tilde{E}| = \beta$ and $\chi_{\tilde{E}} \neq \hat{f}$. Let $v \in H_0^s(\Omega)$ be the weak solution to the following fractional boundary value problem,

$$\begin{cases} (-\Delta)^s v = \chi_{\tilde{E}} & \text{in } \Omega, \\ v = 0 & \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

By the uniqueness of solution, we have $v \neq \hat{u}$. Set $\tilde{u} := \frac{1}{2}\hat{u} + \frac{1}{2}v$. Then, $\tilde{u} \neq \hat{u}$, and $(-\Delta)^s \tilde{u} = \frac{1}{2}\hat{f} + \frac{1}{2}\chi_{\tilde{E}} \in \bar{\mathcal{R}}_\beta$. Now, it suffices to show that

$$[\tilde{u}]_s^2 > [\hat{u}]_s^2,$$

or equivalently

$$\frac{1}{2}[\hat{u}]_s^2 + \frac{1}{2}[v]_s^2 + \iint_{\mathbb{R}^{2n}} \frac{(\hat{u}(x) - \hat{u}(y))\,(v(x) - v(y))}{|x-y|^{n+2s}} dxdy > 2[\hat{u}]_s^2, \qquad (3.9)$$

which would contradict the maximality of $\hat{u}$. But, by elementary computations, (3.9) is equivalent to,

$$\frac{1}{2}\,[\hat{u} - v]_s^2 > 2 \iint_{\mathbb{R}^{2n}} \frac{(\hat{u}(x) - \hat{u}(y))\,((\hat{u} - v)(x) - (\hat{u} - v)(y))}{|x-y|^{n+2s}}\, dxdy. \quad (3.10)$$

Next, from Lemma 2.2.2 and Lemma 3.2.4 , we get

$$\begin{aligned} & \iint_{\mathbb{R}^{2n}} \frac{(\hat{u}(x) - \hat{u}(y))\,((\hat{u} - v)(x) - (\hat{u} - v)(y))}{|x-y|^{n+2s}}\, dxdy \\ = & \int_\Omega \hat{u}\left(\hat{f} - \chi_{\tilde{E}}\right) dx \\ = & \ \alpha \int_{\{\hat{u}=\alpha\}} \left(\hat{f} - \chi_{\tilde{E}}\right) dx \\ = & \ \alpha \int_\Omega \left(\hat{f} - \chi_{\tilde{E}}\right) dx \\ = & \ \alpha(\beta - \beta) = 0. \end{aligned}$$

This completes the proof.

□

Proof of Theorem 3.1.1. The main results follow directly from Lemmas 3.2.1 - 3.2.5. □

Proposition 3.2.1. *For the case $\hat{f} = \chi_{\{\hat{u} > \alpha\}}$, it is in general **not** true that the function $\hat{u}(x)$ minimizes the (non-convex) functional*

$$J(u) = [u]_s^2 - 2\int_\Omega \chi_{\{u>\alpha\}} u dx, \qquad (3.11)$$

over $H_0^s(\Omega)$.

*Similarly, for the case $\hat{f} = \chi_{\{\hat{u} \geq \alpha\}}$, it is in general **not** true that the function $\hat{u}(x)$ minimizes the (non-convex) functional*

$$J(u) = [u]_s^2 - 2\int_\Omega \chi_{\{u \geq \alpha\}} u dx, \qquad (3.12)$$

over $H_0^s(\Omega)$.

Proof. We only present the proof of the first case here while the proof of the second case is similar.

Let us introduce the subset of functions which do not have flat positive components as follows

$$\tilde{H}_0^s(\Omega) = \{u \in H_0^s(\Omega) : |\{\hat{u} = t\}| = 0 \text{ for all } t > 0\}\,.$$

Since $\tilde{H}_0^s(\Omega)$ is dense in $H_0^s(\Omega)$ we can replace $H_0^s(\Omega)$ by $\tilde{H}_0^s(\Omega)$ while we are taking supremum or infimum. Using the fact that for any function $u \in \tilde{H}_0^s(\Omega)$ we can always find a real number α_u such that $|\{u > \alpha_u\}| = \beta$, we obtain

$$\begin{aligned}
\Phi_s(\hat{f}) &= \sup_{f \in \bar{\mathcal{R}}_\beta} \sup_{u \in H_0^s(\Omega)} \left(2 \int_\Omega f u dx - [u]_s^2 \right) \\
&= \sup_{u \in \tilde{H}_0^s(\Omega)} \sup_{f \in \bar{\mathcal{R}}_\beta} \left(2 \int_\Omega f u dx - [u]_s^2 \right) \\
&= \sup_{u \in \tilde{H}_0^s(\Omega)} \left(2 \int_\Omega \chi_{\{u > \alpha_u\}} u dx - [u]_s^2 \right) \\
&= - \inf_{u \in \tilde{H}_0^s(\Omega)} \left([u]_s^2 - 2 \int_\Omega \chi_{\{u > \alpha_u\}} u dx \right),
\end{aligned}$$

which implies that

$$\begin{aligned}
&[\hat{u}]_s^2 - 2 \int_\Omega \hat{f} \hat{u} dx = [\hat{u}]_s^2 - 2 \int_\Omega \chi_{\{\hat{u} > \alpha\}} \hat{u} dx \\
= &\inf_{u \in \tilde{H}_0^s(\Omega)} \left([u]_s^2 - 2 \int_\Omega \chi_{\{u > \alpha_u\}} u dx \right).
\end{aligned}$$

However,

$$[\hat{u}]_s^2 - \int_\Omega \chi_{\{\hat{u} > \alpha\}} \hat{u} dx \neq \inf_{u \in H_0^s(\Omega)} \left([u]_s^2 - 2 \int_\Omega \chi_{\{u > \alpha\}} u dx \right).$$

This can be observed on a classical two-ball example used in various PDEs (in this context see [19], [27]). Consider Ω which consists of two disconnected balls. We can always connect them by a very narrow tube, which would preserve the discussion below unchanged. For small values of β the maximizer of the optimal rearrangement problem will concentrate the set $\{\hat{u} > \alpha\}$ in one of the two balls and keep the function zero in the other ball. On contrast the minimizer of the right hand side can reach a smaller value by "copying" the non-zero function to the ball where $\hat{u}$ is zero.

□

Chapter 4

The Minimization Problem

In this Chapter, we consider the fractional version of the optimal rearrangement minimization problem and show its connection with the stable fractional free boundary problem. Also, we analyse the behaviour of solutions as the fractional parameter s goes to 1.

4.1 Set Up of the Fractional Case

The setting of this fractional problem is the following. Let $0 < s < 1$ be fixed and Ω be a bounded, open domain in $\mathbb{R}^n$ with $C^{1,1}$ boundary. To avoid extra notations, in this chapter we will use u_f to denote the solution to

$$\begin{cases} (-\Delta)^s u_f(x) = f(x) & \text{in } \Omega, \\ u_f = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

where $f \in \bar{\mathcal{R}}_\beta$. Also, we define the functional

$$\Phi_s(f) := [u_f]_s^2,$$

where $[\cdot]$ is the Gagliardo-Nirenberg semi-norm (see Definition 2.2.1).

We have the following existence lemma.

Lemma 4.1.1. *Φ_s is strictly convex and sequentially lower semi-continuous with respect to weak* topology. Also, there exist a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_\beta$ of the functional Φ_s.*

Proof. Direct observation gives that $u_{f_1+f_2} = u_{f_1} + u_{f_2}$ with $f_1, f_2 \in \bar{\mathcal{R}}_\beta$. By the convexity of Gagliardo-Nirenberg semi-norm and strict convexity of quadratic function, we obtain the strict convexity of Φ_s. By Lemma 2.2.3, (2.6) and *Cauchy-Schwartz inequality*, we obtain that

$$[u_f]_s^2 = \int_\Omega f u_f dx \leq \|f\|_2 \|u_f\|_2 \leq C\|f\|_2 \, [u_f]_s \, .$$

Hence, we have $[u_f]_s \leq C\|f\|_2$, which implies that the function $f \mapsto u_f$ is strongly continuous from $L^2(\Omega)$ into $H_0^s(\Omega)$. Thus, the functional Φ_s is strongly continuous from $L^2(\Omega)$ into $\mathbb{R}$. Therefore, [11] implies that Φ_s is weakly lower semi-continuous. According to [28], $f_n \rightharpoonup f$ in $L^2(\Omega)$ if $f_n \overset{*}{\rightharpoonup} f$ in $L^\infty(\Omega)$. We have consequently proven the weak* lower semi-continuity of Φ_s.

Note that Φ_s is non-negative. Then, there exists a sequence $f_n \in \bar{\mathcal{R}}_\beta$ such that

$$\Phi_s(f_n) \to \inf_{f \in \bar{\mathcal{R}}_\beta} \Phi_s(f),$$

as $n \to \infty$. By *Banach-Alaoglu's theorem*, there exists a subsequence (still denoted by f_n) and $\hat{f} \in \bar{\mathcal{R}}_\beta$ so that $f_n \overset{*}{\rightharpoonup} \hat{f}$. Consequently, we obtain

$$\Phi_s(\hat{f}) \leq \liminf \Phi_s(f_n),$$

which implies that $\hat{f}$ is the minimizer.

To finish the proof, we will show the uniqueness by contradiction. Assume that $f_1, f_2 \in \bar{\mathcal{R}}_\beta$ are two minimizers and $f_1 \neq f_2$. Let $g = \lambda f_1 + (1 - \lambda) f_2$. By the convexity of $\bar{\mathcal{R}}_\beta$, $g \in \bar{\mathcal{R}}_\beta$ with any $\lambda \in (0, 1)$. But the strict convexity of Φ_s implies

$$\Phi_s(g) < \lambda \Phi_s(f_1) + (1 - \lambda)\Phi_s(f_2), \tag{4.1}$$

which contradicts the minimization of f_1 and f_2. □

4.2 Main results

Thanks to Lemma 4.1.1, we are ready to present the following main results.

Theorem 4.2.1. *Let Φ_s be in (2.7). There exists a unique minimizer $\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$ such that*

$$\Phi_s(\hat{f}) \leq \Phi_s(f)$$

for any $f \in \bar{\mathcal{R}}_\beta$. Let $\hat{u} = u_{\hat{f}}$ be in Definition 2.2.5. For some $\alpha > 0$, the function $\hat{u}$ satisfies the following conditions

$$\begin{cases} \text{There exists a subset } S \subset \{\hat{u} = \alpha\} \text{ such that } \hat{f} < 1 \text{ a.e. in } S; \\ \{\hat{u} < \alpha\} \subset \{\hat{f} = 1 \text{ a.e.}\}; \\ \hat{f} > 0 \text{ a.e. and } 0 \leq \hat{u} \leq \alpha \text{ in } \Omega. \end{cases} \tag{4.2}$$

Remark 4.2.1. *Observe that this result shows a remarkable difference with the local optimal rearrangement problem, since the optimal configuration $\hat{f}$ for the fractional case is not a characteristic function (see Theorem 1.5.2).*

In order to prove the above theorem, we need several lemmas.

The following will be applied in the first lemma.

Definition 4.2.1. *Let* $\Psi : L^2(\Omega) \to \mathbb{R} \cup \{+\infty, -\infty\}$ *be a convex functional. If* $u \in L^2(\Omega)$ *and* $\Psi(u)$ *finite, the sub-differential* $\partial\Psi(u)$ *of* Ψ *at* u *is defined by*

$$\partial\Psi(u) = \left\{ w \in L^2; \Psi(v) - \Psi(u) \geq \langle v - u, w \rangle, v \in L^2 \right\}.$$

If $\partial\Psi(u) \neq \emptyset$ *then* Ψ *is said to be sub-differentiable at* u*, and the elements of* $\partial\Psi(u)$ *are called sub-gradients of* Ψ *at* u.

The following theorem (see [40]) implies that the sum of sub-gradients of two functionals are identical to the sub-gradient of the sum in some conditions.

Theorem 4.2.2 (Moreau-Rockafellar Theorem)**.** *Let* $f, g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be proper convex functions. Then for every* $x_0 \in \mathbb{R}^n$,

$$\partial f(x_0) + \partial g(x_0) \subset \partial(f + g)(x_0).$$

Moreover, suppose that the interior of domain$(f) \cap$ *domain*(g) *is nonempty. Then for every* $x_0 \in \mathbb{R}^n$,

$$\partial f(x_0) + \partial g(x_0) = \partial(f + g)(x_0).$$

We define $L(f) := \int_\Omega \hat{u} f dx$. Then our first lemma implies that $\hat{f}$ in Theorem 4.2.1 is a minimizer of $L(f)$.

Lemma 4.2.1.

$$\int_\Omega \hat{u}\hat{f} dx \leq \int_\Omega \hat{u} f dx,$$

for any $f \in \bar{\mathcal{R}}_\beta$.

Proof. We define $\Phi : L^2(\Omega) \to \bar{\mathbb{R}}$ as following,

$$\Psi(f) = \Phi_s(f) + \xi_{\bar{\mathcal{R}}_\beta}(f),$$

where

$$\xi_{\bar{\mathcal{R}}_\beta} = \begin{cases} 0, & \text{if } f \in \bar{\mathcal{R}}_\beta; \\ +\infty, & \text{if } f \notin \bar{\mathcal{R}}_\beta. \end{cases}$$

By Lemma 4.1.1, Ψ is strictly convex with respect to all $f \in L^2(\Omega)$ and direct observation implies that $\hat{f}$ minimizes Ψ in $L^2(\Omega)$. By Definition 4.2.1, the sub-differential of the functional Ψ at $\hat{f}$ is

$$\partial\Psi(\hat{f}) = \left\{ g \in L^2(\Omega) : \Psi(f) - \Psi(\hat{f}) \geq \left\langle g, f - \hat{f} \right\rangle, \text{for all } f \in L^2(\Omega) \right\}.$$

Then the function $\tilde{g} = 0$ *a.e.* in Ω is one of the sub-gradients. By (2.1) and (2.6), Φ_s is also sub-differentiable at $\hat{f}$ and the sub-gradient is,

$$\partial\Phi_s(\hat{f}) = \{2\hat{u}\}.$$

Again by Lemma 4.1.1, Φ_s is weak* lower semi-continuous. Thus, we may apply Theorem 4.2.2 to obtain

$$\tilde{g} \in \partial\Psi(\hat{f}) = \partial\Phi_s(\hat{f}) + \partial\xi_{\bar{\mathcal{R}}_\beta}(\hat{f}),$$

which implies that

$$-2\hat{u} \in \partial\xi_{\bar{\mathcal{R}}_\beta}(\hat{f}).$$

But

$$\begin{aligned} & \partial\xi_{\bar{\mathcal{R}}_\beta}(\hat{f}) \\ = & \left\{g \in L^2(\Omega) : \xi_{\bar{\mathcal{R}}_\beta}(f) - \xi_{\bar{\mathcal{R}}_\beta}(\hat{f}) \geq \left\langle g, f - \hat{f}\right\rangle, \text{for all } f \in L^2(\Omega)\right\} \\ = & \left\{g \in L^2(\Omega) : 0 \geq \left\langle g, f - \hat{f}\right\rangle, \text{for all } f \in \bar{\mathcal{R}}_\beta\right\}, \end{aligned}$$

which concludes the proof. □

The following lemma claims that there exists another minimizer of $L(f)$ in $\mathcal{R}_\beta$.

Lemma 4.2.2. *There exists a function $\tilde{f} \in \mathcal{R}_\beta$ such that*

$$\int_\Omega \hat{u}\tilde{f}dx \leq \int_\Omega \hat{u}fdx,$$

for all $f \in \mathcal{R}_\beta$.

Proof. This follows from Lemma 4.2.1, Theorem 1.1.1, Lemma 2.1.1, and the fact that the minimum of the linear functional $L(f)$ on a bounded closed convex set $\bar{\mathcal{R}}_\beta$ is attained in an extreme point $\tilde{f} = \chi_E \in \mathcal{R}_\beta$. □

In the following lemma, we apply Lemma 2.1.3.

Lemma 4.2.3. *There exists $\alpha > 0$ such that any $\tilde{f} = \chi_E$ with E satisfying the condition*

$$\{\hat{u} < \alpha\} \subset E \subset \{\hat{u} \leq \alpha\}, \tag{4.3}$$

is a minimizer of $L(f)$ over $\bar{\mathcal{R}}_\beta$.

Proof. We apply Lemma 2.1.3 and take $h(x) = \hat{u}(x)$ and $G = \beta$. Observe that $\mathcal{C}$ is identical with $\bar{\mathcal{R}}_\beta$. Hence, we take

$$\alpha = \sup\{t : |\{\hat{u} < t\}| \leq \beta\},$$

and the function

$$g(x) = \chi_{\{\hat{u}<\alpha\}} + c\chi_{\{\hat{u}=\alpha\}} \text{ with } c \in [0, 1], \tag{4.4}$$

where

$$c = \frac{\beta - |\{\hat{u} < \alpha\}|}{|\{\hat{u} = \alpha\}|}, \tag{4.5}$$

is a minimizer of $\int_\Omega \hat{u} f dx$ for any $f \in \bar{\mathcal{R}}_\beta$. We claim that any $\tilde{f} = \chi_E$ with E satisfying (4.3) is also a minimizer. Indeed,

$$\begin{aligned}
& \int_\Omega \hat{u}\hat{f}dx - \int_\Omega \hat{u}gdx \\
= & \int_{\{\hat{u}<\alpha\}} \hat{u}dx + \int_{E\setminus\{\hat{u}<\alpha\}} \alpha dx - \int_{\{\hat{u}<\alpha\}} \hat{u}dx - \int_{\{\hat{u}=\alpha\}} c\alpha dx \\
= & \ 0 \qquad \text{by (4.5).}
\end{aligned}$$

□

In the proof of Lemma 4.2.3, we see that $|\{\hat{u} < \alpha\}| \leq \beta$. Thus, we utilize some basic techniques in rearrangement and obtain the following.

Lemma 4.2.4.

$$\hat{f} = 1 \ a.e. \ in \ \{\hat{u} < \alpha\}.$$

Proof. Assume that there is $A \subset \{\hat{u} < \alpha\}$ with positive measure such that $\hat{f} < 1$ in A. Then $\hat{f} > 0$ in some subset of $\{\hat{u} \geq \alpha\}$. we may replace $\hat{f}$ by a function $f \in \bar{\mathcal{R}}_\beta$ which has larger values in A and smaller values in $\{\hat{u} \geq \alpha\}$. Consequently,

$$\int_\Omega f\hat{u}dx < \int_\Omega \hat{f}\hat{u}dx,$$

which contradicts the minimality of $\hat{f}$. □

Lemma 4.2.5.

$$\{\hat{u} > \alpha\} \subset \{\hat{f} = 0 \ a.e.\}.$$

Proof. Recall that $\hat{f}, \tilde{f} \in \bar{\mathcal{R}}_\beta$. we obtain

$$\begin{aligned}
\beta &= \int_\Omega \hat{f}dx \\
&= \int_{\{\hat{u}<\alpha\}} \hat{f}dx + \int_{\{\hat{u}=\alpha\}} \hat{f}dx + \int_{\{\hat{u}>\alpha\}} \hat{f}dx = \int_\Omega \tilde{f}dx \\
&= \int_{\{\hat{u}<\alpha\}} \tilde{f}dx + \int_{\{\hat{u}=\alpha\}} \tilde{f}dx + \int_{\{\hat{u}>\alpha\}} \tilde{f}dx.
\end{aligned}$$

By Lemma 4.2.3 and 4.2.4, we obtain

$$\int_{\{\hat{u}=\alpha\}} \tilde{f}dx = \int_{\{\hat{u}=\alpha\}} \hat{f}dx + \int_{\{\hat{u}>\alpha\}} \hat{f}dx. \tag{4.6}$$

By Lemma 4.2.1 and 4.2.2, we obtain

$$\int_\Omega \hat{u}\hat{f}dx = \int_\Omega \hat{u}\tilde{f}dx. \tag{4.7}$$

Then, Lemma 4.2.3 , 4.2.4 and (4.7) imply

$$\int_{\{\hat{u}\geq\alpha\}} \hat{u}\hat{f}dx = \int_{\{\hat{u}=\alpha\}} \hat{u}\tilde{f}dx. \tag{4.8}$$

Therefore, (4.6) and (4.8) together imply

$$\begin{aligned}
\alpha\int_{\{\hat{u}=\alpha\}} \tilde{f}dx &= \alpha\int_{\{\hat{u}=\alpha\}} \hat{f}dx + \alpha\int_{\{\hat{u}>\alpha\}} \hat{f}dx \\
&\leq \int_{\{\hat{u}=\alpha\}} \hat{u}\hat{f}dx + \int_{\{\hat{u}>\alpha\}} \hat{u}\hat{f}dx = \int_{\{\hat{u}=\alpha\}} \hat{u}\hat{f}dx \\
&= \int_{\{\hat{u}=\alpha\}} \hat{u}\tilde{f}dx = \alpha\int_{\{\hat{u}=\alpha\}} \tilde{f}dx,
\end{aligned}$$

which implies

$$\alpha\int_{\{\hat{u}>\alpha\}} \hat{f}dx = \int_{\{\hat{u}>\alpha\}} \hat{u}\hat{f}dx,$$

as required. □

Here we are ready to prove that the upper barrier of u is α.

Lemma 4.2.6.

$$\{\hat{u} > \alpha\} = \emptyset.$$

Proof. For any $\beta > \alpha$, we take

$$\varphi(x) := (\hat{u}(x) - \beta)^+ .$$

It is easy to verify that $\varphi(x) \in H_0^s(\Omega)$. Now, take $\omega := \operatorname{supp}\varphi(x)$ and observe that $\omega \subset \{\hat{u} > \alpha\}$. In this case, Lemma 4.2.5 implies that

$$\begin{aligned}
0 &= \langle(-\Delta)^s\hat{u}, \varphi\rangle \\
&= \iint_{\mathbb{R}^{n\times n}} \frac{(\hat{u}(x)-\hat{u}(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\
&= \int_\omega \left(\int_\omega \frac{(\hat{u}(x)-\hat{u}(y))^2}{|x-y|^{n+2s}}dy\right) dx \\
&+ \int_\omega \left(\int_{\mathbb{R}^n\setminus\omega} \frac{(\hat{u}(x)-\hat{u}(y))\,(\hat{u}(x)-\beta)}{|x-y|^{n+2s}}dy\right) dx \\
&+ \int_{\mathbb{R}^n\setminus\omega} \left(\int_\omega \frac{(\hat{u}(x)-\hat{u}(y))\,(\beta-\hat{u}(y))}{|x-y|^{n+2s}}dy\right) dx \\
&+ \int_{\mathbb{R}^n\setminus\omega} \left(\int_{\mathbb{R}^n\setminus\omega} \frac{(\hat{u}(x)-\hat{u}(y))\,(0-0)}{|x-y|^{n+2s}}dy\right) dx,
\end{aligned} \tag{4.9}$$

where the last equality is obtained by the setting of φ. On the right hand side of (4.9), the first three integrals are non-negative and the forth integral vanishes. Since $\hat{u}$ is continuous in $\mathbb{R}^n$, the equality holds if and only if

$$|\omega| = |\{\hat{u} > \beta\}| = 0,$$

which completes the proof. $\square$

The following lemma implies that $|\{0 < \hat{f} < 1\}| > 0$, thus we proved that $\hat{f} \notin \mathcal{R}_\beta$.

Lemma 4.2.7.

$$\left|\{\hat{f} = 0\}\right| = 0.$$

Proof. Note that $\hat{f} \in \bar{\mathcal{R}}_\beta$ implies $\hat{f} \geq 0$. By Lemma 4.2.4 and Lemma 4.2.6, it suffices to check that $\hat{f} > 0$ pointwise in the subset $\{\hat{u} = \alpha\}$. By the singular integral definition of $(-\Delta)^s \hat{u}$ and Lemma 4.2.6, we obtain the following for any point in $\{\hat{u} = \alpha\}$,

$$\begin{aligned} \hat{f}(x) &= (-\Delta)^s \hat{u}(x) \\ &= \text{p.v.} \int_{\mathbb{R}^n} \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^{n+2s}} dy \\ &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{\hat{u}(x) - \hat{u}(y)}{|x-y|^{n+2s}} dy \\ &> \int_{\mathbb{R}^n \setminus \Omega} \frac{\alpha}{|x-y|^{n+2s}} dy > 0, \end{aligned}$$

as required. $\square$

Now, we are at the point to prove the main result.

Proof of Theorem 4.2.1. Lemma 4.2.4 and Lemma 4.2.6 imply that there exists a subset $S \subset \{\hat{u} = \alpha\}$ with $|S| > 0$ such that $\hat{f} < 1$ *a.e.* in S, which is the first condition in (5.3.1); Lemma 4.2.4 implies the second condition; The third condition is obtained directly by Lemma 4.2.6 and 4.2.7. Finally, 4.2.7 implies that the minimizer $\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$. $\square$

4.3 The Normalized Fractional Obstacle Problem

We divide this section into two subsections. The first subsection is devoted to the study of the connection between the solutions to the optimal fractional rearrangement problem considered in Section 4.2 and solutions of the normalized fractional obstacle problem. We find the corresponding fractional analogue of (1.23) and prove that the solution of the fractional rearrangement problem is a solution to the fractional normalized obstacle problem.

4.3.1 The Fractional Obstacle Functional

Our first result is the following.

Theorem 4.3.1. *Let $\hat{f} \in \bar{\mathcal{R}}_\beta$ be the solution to the optimal rearrangement problem and $\hat{u} := u_{\hat{f}} \in H_0^s(\Omega)$ be given by* (2.5)*. Let $\alpha > 0$ be the constant given in Theorem 4.2.1. We define*

$$\hat{U} := \alpha - \hat{u}.$$

Then $\hat{U}$ is the minimizer of the following functional

$$J(v) = \frac{1}{2}[v]_s^2 + \int_\Omega v(x)^+ dx \tag{4.10}$$

among the family of functions

$$H_\alpha(\Omega) := \{v : \alpha - v \in H_0^s(\Omega)\}.$$

Moreover, $\hat{U}$ verifies

$$\chi_{\{U>0\}} \leq -(-\Delta)^s \hat{U} \leq \chi_{\{U\geq 0\}} \quad \text{in } \Omega \tag{4.11}$$

in sense of distribution, i.e.,

$$\int_\Omega \chi_{\{U>0\}}(x)\varphi(x)dx \leq \left\langle -(-\Delta)^s \hat{U}, \varphi \right\rangle \leq \int_\Omega \chi_{\{U\geq 0\}}(x)\varphi(x)dx$$

for all $\varphi \in H_0^s(\Omega)$. Also, the minimizer of J is unique and it is the unique solution to the inequality (4.11)*.*

Proof. We define

$$I(v) := \frac{1}{2}[v]_s^2 + \int_\Omega \hat{f} v dx,$$

where $v \in H_\alpha(\Omega)$. Since $\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$, we have $I(v) \leq J(v)$ for any $v \in H_\alpha(\Omega)$. Therefore, we apply Theorem 4.2.1 and obtain

$$\int_\Omega \hat{U}^+ dx = \int_\Omega \hat{U} dx = \int_\Omega \hat{f}\hat{U} dx,$$

which implies that $I(\hat{U}) = J(\hat{U})$. Now, we claim that $\hat{U}$ is the minimizer of I. The existence of minimizer of I in $H_\alpha(\Omega)$ follows from the convexity of I. We say that $v \in H_\alpha(\Omega)$ is a minimizer of I if

$$I(v) \leq I(v + \varepsilon\varphi), \text{for any } \varepsilon \in \mathbb{R}, \varphi \in H_0^s(\Omega). \tag{4.12}$$

By expanding (4.12), we obtain the following sufficient condition for a minimizer

$$\iint_{\mathbb{R}^{n\times n}} \frac{(v(x) - v(y)) \cdot (\varphi(x) - \varphi(y))}{|x-y|^{n+2s}} dxdy = -\int_\Omega \hat{f}\varphi dx, \tag{4.13}$$

for any $\varphi \in H_0^s(\Omega)$. By Definition 2.2.5, $\hat{U}$ trivially satisfies (4.13), which proves the claim. In this case, we have

$$J(v) \geq I(v) \geq I(\hat{U}) = J(\hat{U}), \quad \text{for any } v \in H_\alpha(\Omega),$$

which proved that $\hat{U}$ is also the minimizer of J. Now, we take the variation $U_\varepsilon(x) = \hat{U}(x) + \varepsilon\varphi(x)$ with $\varphi \in C_0^\infty(\Omega), \varepsilon \in \mathbb{R}$. Thus $J(U_\varepsilon) \geq J(\hat{U})$ implies

$$\frac{1}{2}\varepsilon^2[\varphi]_s^2 + \varepsilon\left\langle(-\Delta)^s\hat{U}, \varphi\right\rangle + \int_\Omega \left(\hat{U} + \varepsilon\varphi\right)^+ dx - \int_\Omega \hat{U}^+ \geq 0. \tag{4.14}$$

Therefore, (4.14) implies (4.11) as required. Also, the uniqueness of minimizer of functional J is a immediate consequence of the strict convexity of J. Finally, assume that $U \in H_\alpha(\Omega)$ satisfies (4.11) and $U \neq \hat{U}$. Then, $J(\hat{U}) < J(U)$. We define $U_\varepsilon := U + \varepsilon(\hat{U} - U)$ with $\varepsilon \in \mathbb{R}$, and observe that $U_\varepsilon \in H_\alpha(\Omega)$. Since functional J is strictly convex, we have

$$\lim_{\varepsilon\to 0^+} \frac{J(U_\varepsilon) - J(U)}{\varepsilon} < 0. \tag{4.15}$$

Now, we take $\Psi := \hat{U} - U \in H_0^s(\Omega)$ and use the decomposition $\Psi = \Psi^+ - \Psi^-$ in (4.15). Therefore

$$\begin{aligned}
& 0 > \lim_{\varepsilon\to 0^+} \frac{J(U_\varepsilon) - J(U)}{\varepsilon} \\
= \ & \frac{1}{\varepsilon}\left(\frac{1}{2}[U + \varepsilon\Psi]_s^2 + \int_\Omega (U + \varepsilon\Psi)^+ dx - \frac{1}{2}[U]_s^2 - \int_\Omega U^+ dx\right) \\
= \ & \iint_{\mathbb{R}^{n\times n}} \frac{(U(x) - U(y))(\Psi(x) - \Psi(y))}{|x-y|^{n+2s}} dxdy \\
+ \ & \int_\Omega \chi_{\{U>0\}}\Psi + \chi_{\{U=0\}}\Psi^+ dx \\
= \ & \iint_{\mathbb{R}^{n\times n}} \frac{(U(x) - U(y))(\Psi^+(x) - \Psi^+(y))}{|x-y|^{n+2s}} dxdy \\
+ \ & \int_\Omega \chi_{\{U\geq 0\}}\Psi^+ dx \\
- \ & \iint_{\mathbb{R}^{n\times n}} \frac{(U(x) - U(y))(\Psi^-(x) - \Psi^-(y))}{|x-y|^{n+2s}} dxdy \\
- \ & \int_\Omega \chi_{\{U>0\}}\Psi^- dx \\
\geq \ & 0, \qquad \text{by assumption.}
\end{aligned}$$

Consequently, we reach a contradiction and thus we have proven that $\hat{U}$ is the unique solution to (4.11). □

Remark 4.3.1. *This result again shows an interesting difference between the classical obstacle problem and the fractional normalized version. Observe* *that in the positivity set, we still have $-(-\Delta)^s\hat{U} = 1$, but in the zero set of $\hat{U}$, the function $\hat{U}$ is not s-harmonic (even if it is identically zero!). The free boundary condition on $\partial\{\hat{U} > 0\}$ is given by the fact that $(-\Delta)^s\hat{U}$ is a function bounded by 0 and 1 across the free boundary.*

4.3.2 The Nonlocal Normalized Obstacle Equation

The results in Theorem 4.3.1 are not completely satisfactory, since we do not obtain an equation satisfied by $\hat{U}$ but only the inequalities (4.11). In this subsection, our last result shows that in fact $\hat{U}$ is the solution to a fully nonlinear equation.

Theorem 4.3.2. *Let $\hat{U}$ be the solution of the normalized fractional obstacle problem given by Theorem 4.3.1. Then, $\hat{U}$ is a solution to*

$$\begin{cases} -(-\Delta)^s U - \chi_{\{U\leq 0\}} \min\left\{-(-\Delta)^s U^+; 1\right\} = \chi_{\{U>0\}}, & \text{in } \Omega, \\ U = \alpha, & \text{in } \Omega^c. \end{cases} \tag{4.16}$$

Moreover, the equation in (4.16) *is equivalent to* (4.11). *Finally, U verifies* (4.16) *if and only if it is a minimizer of J in H_α, where J and H_α are given in Theorem 4.3.1.*

Before we start the proof, let us observe that for any $u \in H^s(\mathbb{R}^n)$, we have

$$[u^\pm]_s \leq [u]_s$$

and thus $(-\Delta)^s u^\pm \in H^{-s}(\mathbb{R}^n)$. On the other hand, $(-\Delta)^s u^+$ is a distribution and the expression

$$\min\left\{-(-\Delta)^s U^+; 1\right\} = -\max\left\{(-\Delta)^s U^+; -1\right\} = 1 - \left((-\Delta)^s U^+ + 1\right)^+$$

makes in general no sense, unless $(-\Delta)^s U^+$ is a signed measure in Ω (See Theorem 6.22 in [41]). Indeed, the following statement implies that $\chi_{\{U\leq 0\}} \cdot (-(-\Delta)^s U^+)$ is a positive distribution.

Lemma 4.3.1. *If $U \in H_\alpha$, then $\chi_{\{U\leq 0\}} \cdot (-\Delta)^s U^+ \leq 0$.*

Proof. For any $\varphi \in H^s_0(\Omega)$ such that $\varphi(x) \geq 0$ for all x, we have

$$\begin{aligned} &\left\langle (-\Delta)^s U^+, \chi_{\{U\leq 0\}} \cdot \varphi \right\rangle \\ = &\iint_{\mathbb{R}^{n\times n}} \frac{(U^+(x) - U^+(y)) \left(\chi_{\{U\leq 0\}}\varphi(x) - \chi_{\{U\leq 0\}}\varphi(y)\right)}{|x-y|^{n+2s}} dxdy \\ = &\int_{\{U\leq 0\}} \int_{\{U>0\}} \frac{U^+(x) \cdot -\varphi(y)}{|x-y|^{n+2s}} dxdy + \\ &\int_{\{U>0\}} \int_{\{U\leq 0\}} \frac{-U^+(y) \cdot \varphi(x)}{|x-y|^{n+2s}} dxdy \leq 0, \end{aligned}$$

as required. □

Therefore, we need to search for solutions of (4.16) only among functions U such that $(-\Delta)^s U \leq 0$ in Ω. This leads us to the introduction of *fractional subharmonic functions* in Ω, which form a convex subset of $H^s(\mathbb{R}^n)$, i.e.,

$$H^s_{sub}(\Omega) := \{U - \alpha \in H^s(\mathbb{R}^n) : (-\Delta)^s U \leq 0 \text{ in } \Omega\}.$$

The following lemma is essential for equation (4.16) to make sense.

Lemma 4.3.2. *If $U \in H^s_{sub}(\Omega)$, then $U^+ \in H^s_{sub}(\Omega)$.*

Proof. Assume U is smooth, and then its fractional Laplacian has pointwise values. Direct calculation gives the following.

1. For $x \in \{U \leq 0\}$,

$$(-\Delta)^s U^+(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{-U^+(y)}{|x-y|^{n+2s}} dy \leq 0.$$

2. For $x \in \{U > 0\}$,

$$\begin{aligned} (-\Delta)^s U^+(x) &= \text{p.v.} \int_{\mathbb{R}^n} \frac{U(x) - U^+(y)}{|x-y|^{n+2s}} dy \\ = \quad & \text{p.v.} \int_{\mathbb{R}^n} \frac{U(x) - U(y)}{|x-y|^{n+2s}} dy \\ & -\text{p.v.} \int_{\mathbb{R}^n} \frac{U^-(y)}{|x-y|^{n+2s}} dy \leq 0. \end{aligned}$$

For a general $U \in H^s(\mathbb{R}^n)$, observe that U is locally integrable and thus we define its mollification $U_\varepsilon := \eta_\varepsilon * U$, where η_ε is a family of smooth approximation functions of the identity such that $\eta_\varepsilon(x) = \eta_\varepsilon(|x|)$. Thus, it suffices to show that

$$\langle (-\Delta)^s U_\varepsilon, \varphi \rangle = \langle (-\Delta)^s U, \varphi_\varepsilon \rangle, \tag{4.17}$$

for every $\varphi \in C_0^\infty(\mathbb{R}^n)$. Indeed, we assume that (4.17) holds. If $(-\Delta)^s U \leq 0$, then $(-\Delta)^s U_\varepsilon \leq 0$ for every $\varepsilon > 0$. From above, we have $(-\Delta)^s (U_\varepsilon)^+ \leq 0$. Since $(U_\varepsilon)^+ \to U^+$ in $L^2(\mathbb{R}^n)$ as $\varepsilon \to \infty$ and therefore $(-\Delta)^s U^+ \to (-\Delta)^s (U_\varepsilon)^+$ in the sense of distribution. In order to prove (4.17), we may introduce

$$D^s U(x,y) = \frac{U(x) - U(y)}{|x-y|^{\frac{n}{2}+s}},$$

which is the Hölder quotient of order s of U. Then, we observe that

$$\begin{aligned}
& \langle(-\Delta)^s U_\varepsilon, \varphi\rangle \\
= & \iint_{\mathbb{R}^{n\times n}} D^s U_\varepsilon(x,y) D^s\varphi(x,y)dxdy \\
= & \iint_{\mathbb{R}^{n\times n}} \int_{B(0,\varepsilon)} D^s U(x-z, y-z)\eta_\varepsilon(z) D^s\varphi(x,y)dzdxdy \\
= & \iint_{\mathbb{R}^{n\times n}} \int_{B(0,\varepsilon)} D^s U(x,y)\eta_\varepsilon(z) D^s\varphi(x+z,y+z)dzdxdy \\
= & \iint_{\mathbb{R}^{n\times n}} \int_{B(0,\varepsilon)} D^s U(x,y)\eta_\varepsilon(z) D^s\varphi(x-z,y-z)dzdxdy \\
= & \iint_{\mathbb{R}^{n\times n}} D^s U(x,y) D^s\varphi_\varepsilon(x,y)dxdy \\
= & \langle(-\Delta)^s U, \varphi_\varepsilon\rangle .
\end{aligned}$$

The proof is complete. □

Corollary 4.3.1. *If $U \in H^s_{sub}(\Omega)$, then*

$$\min\left\{-(-\Delta)^s U^+; 1\right\} \in L^\infty(\Omega).$$

In this case, for $\alpha > 0$, we can formulate the following normalized fractional obstacle problem,

$$-(-\Delta)^s U - \chi_{\{U\le 0\}} \min\left\{-(-\Delta)^s U^+; 1\right\} = \chi_{\{U>0\}}, \tag{4.18}$$

among continuous functions $U \in H^s_{sub}(\Omega)$ such that $U = \alpha$ in $\mathbb{R}^n \setminus \Omega$. Also, the weak formulation of the equation (4.18) is

$$\begin{aligned}
& -\iint_{\mathbb{R}^{n\times n}} \frac{(U(x)-U(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\
= & \int_\Omega \chi_{\{U>0\}}(x)\varphi(x)dx \\
+ & \min\left\{\left\langle -(-\Delta)^s U^+, \chi_{\{U\le 0\}}\cdot\varphi\right\rangle ; \int_\Omega \chi_{\{U>0\}}(x)\varphi(x)dx\right\},
\end{aligned}$$

for any $\varphi \in H^s_0(\Omega)$.

With the preparations above, we are in the position to finish this section by proving Theorem 4.3.2.

Proof of Theorem 4.3.2. We only need to show that (4.18) and (4.11) are equivalent when (4.18) makes sense. We may break down the proof into several claims.

Claim 4.3.1. *If U is the solution to* (4.18)*, then $U \ge 0$.*

Proof. First observe that

$$\begin{aligned} & (U(x)-U(y))\cdot\left(U^-(x)-U^-(y)\right) \\ = \; & \left(U^+(x)-U^+(y)\right)\cdot\left(U^-(x)-U^-(y)\right)-\left(U^-(x)-U^-(y)\right)^2. \end{aligned}$$

The identity above directly implies that

$$\langle(-\Delta)^s U, U^-\rangle = \langle(-\Delta)^s U^+, U^-\rangle - [U^-]_s^2. \tag{4.19}$$

Since $U^- \in H_0^s(\Omega)$, we may take it as a test function in the weak formulation of (4.18).

$$\begin{aligned} & \left\langle(-\Delta)^s U, U^-\right\rangle \\ = \; & -\left\langle \min\{-(-\Delta)^s U^+;1\}, \chi_{\{U\le 0\}}\cdot U^-\right\rangle - 0 \qquad (4.20) \\ = \; & -\left\langle \min\{-(-\Delta)^s U^+;1\}, U^-\right\rangle \\ = \; & \left\langle \max\{-(-\Delta)^s U^+;1\}, U^-\right\rangle \\ \ge \; & \left\langle(-\Delta)^s U^+, U^-\right\rangle. \qquad (4.21) \end{aligned}$$

Therefore, by (4.19) and (4.20), we arrive at $[U^-]_s^2 \le 0$. This implies that $\{U^- > 0\} = \emptyset$, as required. □

Claim 4.3.2. (4.18) *implies* (4.11).

Proof. It is immediately from Lemma 4.3.2 and Claim 4.3.1. □

Claim 4.3.3. (4.11) *implies* $U \ge 0$.

Proof. Let U be a solution to (4.11). For any $\beta < 0$, we take

$$\varphi(x) := (U(x)-\beta)^-.$$

Thus, $\omega := \operatorname{supp}\varphi \subset \{U<0\}$. Then,

$$\begin{aligned} 0 \;=\; & \langle(-\Delta)^s U, \varphi\rangle \qquad (4.22) \\ =\; & \iint_{\mathbb{R}^{n\times n}} \frac{(U(x)-U(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \\ =\; & \int_\omega\left(\int_\omega \frac{(U(x)-U(y))\,(U(y)-U(x))}{|x-y|^{n+2s}}dy\right)dx \\ +\; & \int_\omega\left(\int_{\mathbb{R}^n\setminus\omega} \frac{(U(x)-U(y))\,(\beta-U(x))}{|x-y|^{n+2s}}dy\right)dx \\ +\; & \int_{\mathbb{R}^n\setminus\omega}\left(\int_\omega \frac{(U(x)-U(y))\,(U(y)-\beta)}{|x-y|^{n+2s}}dy\right)dx \\ +\; & \int_{\mathbb{R}^n\setminus\omega}\left(\int_{\mathbb{R}^n\setminus\omega} \frac{(U(x)-U(y))\,(0-0)}{|x-y|^{n+2s}}dy\right)dx, \qquad (4.23) \end{aligned}$$

where the last equation is obtained by the definition of φ. On the right hand side of (4.22), the first three integrals are non-negative. Since U is continuous in $\mathbb{R}^n$, the equality holds if and only if

$$|\omega| = |\{u < \beta\}| = 0,$$

which completes the proof. □

Claim 4.3.4. (4.11) *implies* (4.18).

Proof. This can be verified directly. □

The proof of the theorem is complete. □

4.4 The Behaviour of The Optimal Rearrangement Problem as $s \to 1$

4.4.1 The constant $C(n,s)$

We introduce the normalizing constant which only depends on n and s as following

$$C(n,s) = \left(\int_{\mathbb{R}^n} \frac{1 - \cos(\zeta_1)}{|\zeta|^{n+2s}} d\zeta \right)^{-1}, \tag{4.24}$$

where ζ_1 is the first coordinate of ζ. Now, we modify the definition of Gagliardo-Nirenberg seminorm and fractional Laplacian in Definition 2.2.1 and 2.2.2,

$$[v]_s^2 = C(n,s) \iint_{\mathbb{R}^{n\times n}} \frac{(v(x) - v(y))^2}{|x-y|^{n+2s}} dxdy, \tag{4.25}$$

and

$$(-\Delta)^s u(x) = \text{p.v. } C(n,s) \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy. \tag{4.26}$$

The following comes from [30], Corollary 4.2.

Corollary 4.4.1. *For any $n > 1$, the following statements hold.*

1. $$\lim_{s \to 1^-} \frac{C(n,s)}{s(1-s)} = \frac{4n}{\omega_{n-1}};$$

2. $$\lim_{s \to 0^+} \frac{C(n,s)}{s(1-s)} = \frac{2}{\omega_{n-1}},$$

where ω_{n-1} is the $n-1$ dimensional measure of $\mathbb{S}^{n-1}$.

Moreover, we have the following result.

Proposition 4.4.1. *Let $u \in L^2(\Omega)$ and $u = 0 \in \mathbb{R}^n \setminus \Omega$ be fixed. When $s_k \to 1^-$ as $k \to \infty$, we have*

$$\begin{cases} \lim_{k\to\infty}[u]_{s_k}^2 = C(n)\|\nabla u\|_2^2 & \text{if } u \in W^{1,2}(\mathbb{R}^n), \\ \liminf_{k\to\infty}[u]_{s_k}^2 = \infty & \text{if } u \notin W^{1,2}(\mathbb{R}^n), \end{cases}$$

where

$$C(n) = 2n\frac{1}{\omega_{n-1}}\int_{\omega\in\mathbb{S}^{n-1}} |\omega\cdot\hat{e}|^2\, d\sigma, \quad \text{for any unit vector } \hat{e} \text{ in } \mathbb{R}^n. \tag{4.27}$$

Also, if $u \in W^{1,2}(\mathbb{R}^n)$, $(-\Delta)^s u \to -\Delta u$ in the sense of distributions, i.e.,

$$C(n,s)\iint_{\mathbb{R}^{n\times n}} \frac{(u(x)-u(y))\,(\varphi(x)-\varphi(y))}{|x-y|^{n+2s}}dxdy \to \int_{\mathbb{R}^n} \langle\nabla u, \nabla\varphi\rangle\, dx, \tag{4.28}$$

for any $\varphi \in W_0^{1,2}(\mathbb{R}^n)$.

Proof. According to [30, Proposition 2.1], $u \in W^{1,2}(\mathbb{R}^n)$ if $\liminf_{k\to\infty}[u]_{s_k}^2 < \infty$. Let $R > 0$ be so that $\Omega \subset B(R)$ and $\bar{R} = R + 1$. Observe that for all $u \in W^{1,2}(\mathbb{R}^n)$ and $h \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |u(x+h)-u(x)|^2\, dx \le |h|\int_{\mathbb{R}^n} |\nabla u|^2\, dx. \tag{4.29}$$

Also, we obtain

$$\begin{aligned} [u]_{s_k}^2 &= C(n,s_k)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(\bar{R})} \frac{|u(y)|^2}{|x-y|^{n+2s_k}}dxdy \\ &+ \quad C(n,s_k)\int_{\mathbb{R}^n\setminus B(\bar{R})}\int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x-y|^{n+2s_k}}dxdy \\ &+ \quad C(n,s_k)\int_{B(\bar{R})}\int_{B(\bar{R})} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_k}}dxdy + 0 \end{aligned} \tag{4.30}$$

We apply (4.29) and Corollary 4.4.1 to the third term of (4.30) and obtain

$$\begin{aligned} & C(n,s_k)\int_{B(\bar{R})}\int_{B(\bar{R})} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_k}}dxdy \\ \le\ & C(n,s_k)\int_{B(2\bar{R})} \frac{1}{|h|^{n+2s_k}}\left(\int_{B(\bar{R})} |u(x+h)-u(x)|^2 dx\right) dh \\ \le\ & C(n,s_k)\,\|\nabla u\|_2^2\cdot\omega_{n-1}\int_0^{2R} \frac{1}{r^{n+2s_k-2}}\cdot r^{n-1}dr \\ =\ & 2ns_k\,\|\nabla u\|_2^2\,(2R)^{2-2s_k} \to 2n\cdot\|\nabla u\|_2^2 \quad \text{as } k\to\infty. \end{aligned} \tag{4.31}$$

The first term and second term of (4.30) coincide, and reasoning as before we get

$$\begin{aligned} & C(n,s_k)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n\setminus B(\bar{R})}\frac{|u(y)|^2}{|x-y|^{n+2s_k}}dxdy \\ \leq\ & 2\|u\|_2^2\cdot n(1-s_k)\to 0 \ \text{ as } k\to\infty. \end{aligned} \tag{4.32}$$

Therefore, (4.30), (4.31) and (4.32) together imply that $[u]_{s_k}^2 \leq 2n\|\nabla u\|_2^2$ as $k\to\infty$. For any $u,v\in W_0^{1,2}(\Omega)$, let

$$U_k(x,y) = (C(n,s_k))^{1/2}\cdot\frac{|u(x)-u(y)|^{1/2}}{|x-y|^{\frac{1}{2}n+s_k}}.$$

Thus, as $k\to\infty$

$$|\|U_k\|_{L^2}-\|V_k\|_{L^2}| \leq \|U_k-V_k\|_{L^2} \leq 2n\|\nabla(u-v)\|_2^2.$$

For some dense subset of $W^{1,2}(\Omega)$, e.g., $u\in C^2(\bar{\Omega})$ and $u=0\in\mathbb{R}^n\setminus\bar{\Omega}$, observe that

$$\frac{|u(x)-u(y)|}{|x-y|} = \frac{|\nabla u(x)\cdot(x-y)|}{|x-y|} + O(|x-y|). \tag{4.33}$$

For any fixed $x\in\mathbb{R}^n$, let $R>0$ be so that $\Omega\subset B(x,R)$. Then,

$$\begin{aligned} & C(n,s_k)\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_k}}dy \\ =\ & C(n,s_k)\int_{B(x,R)}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s_k}}dy \\ +\ & C(n,s_k)\int_{\mathbb{R}^n\setminus B(x,R)}\frac{|u(x)|^2}{|x-y|^{n+2s_k}}dy. \end{aligned} \tag{4.34}$$

By Corollary 4.4.1, the second term of (4.34) vanishes as $k\to\infty$. Then, applying (4.33) we have

$$\begin{aligned} & \text{Left hand side of (4.34)} \\ =\ & C(n,s_k)\int_{B(x,R)}\frac{1}{|x-y|^{n+2s_k-2}}\cdot\frac{|u(x)-u(y)|^2}{|x-y|^2}dy && (4.35) \\ =\ & C(n,s_k)\int_0^R\frac{1}{r^{n+2s_k-2}}\int_{|x-y|=r}\left|\nabla u(x)\cdot\frac{x-y}{|x-y|}\right|^2 && (4.36) \\ & +O(|x-y|^2)d\sigma dr \\ =\ & C(n,s_k)\int_0^R\frac{1}{r^{n+2s_k-2}}\int_{|\omega|=r}\left|\nabla u(x)\cdot\frac{\omega}{|\omega|}\right|^2+O(r^2)d\sigma dr \\ =\ & K\,|\nabla u(x)|^2\,\frac{\omega_{n-1}C(n,s_k)}{2(1-s_k)}R^{2(1-s_k)}+\frac{\omega_{n-1}C(n,s_k)}{4-s_k}O(R^{4-s_k}) \\ \to\ & 2nK\,|\nabla u(x)|^2, \quad \text{as } k\to\infty, && (4.37) \end{aligned}$$

where

$$K = \frac{1}{\omega_{n-1}} \int_{\omega \in \mathbb{S}^{n-1}} |\omega \cdot \hat{e}|^2 \, d\sigma, \quad \text{for any unit vector } \hat{e}.$$

Since $u(x)$ is *Lipschitz continuous* in $\bar{\Omega}$, it is easy to verify that

$$\int_{\mathbb{R}^n} (U_k(x,y))^2 dy$$

is bounded for any $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$. By *dominated convergence*, (4.34) implies

$$\lim_{k\to\infty} \|U_k(x,y)\|_2^2 = 2nK \|\nabla u\|_2^2,$$

as required. To prove the second part of the theorem, we note that $C^\infty(\bar{\Omega})$ is dense in $W^{1,2}(\Omega)$. Then, it suffices to show that $\lim_{k\to\infty} (-\Delta)^{s_k} v = -\Delta v$ for any $v \in C_c^\infty(\mathbb{R}^n)$, which is proven in [30, Proposition 4.4]. □

Moreover, we have the following stronger statement.

Proposition 4.4.2. *Assume $s_k \to 1^-$ as $k \to \infty$ and $\{u_k\} \in L^2(\Omega)$, $u_k = 0$ in $\mathbb{R}^n \setminus \Omega$. Also, assume that*

$$\sup_k \|u_k\|_{L^2} < \infty \quad \text{and} \quad \sup_k [u_k]_{s_k} < \infty.$$

Then, there exists a function $u \in W_0^{1,2}(\Omega)$ such that (up to a subsequence)

$$u_k \to u \quad \text{in} \quad L^2_{loc}(\mathbb{R}^n) \quad \text{and} \quad C(n)\|\nabla u\|_{L^2}^2 \le \liminf_{k\to\infty} [u_k]_{s_k}^2,$$

where $C(n)$ is in (4.27).

Proof. Lemma 2.2.4 and Proposition 4.4.1 together give the required results. □

4.4.2 The Limit $s \to 1^-$

Now, we go back to the *Fractional Optimal Rearrangement Problem.* Let the sequence $\{s_k\} \in (0,1)$ be such that $s_k \to 1^-$ as $k \to \infty$. Let $[\cdot]_{s_k}^2$ be in (4.25) and $(-\Delta)^{s_k}$ be in (4.26).

Let $u_k = u_{f_k}$ be the solution to the following *modified fractional boundary value problem*

$$\begin{cases} (-\Delta)^{s_k} u_k(x) = f_k(x) \text{ in } \Omega, \\ u_k(x) = 0 \qquad\qquad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases}$$

where

$$f_k \in \bar{\mathcal{R}}_{\tilde{\beta}} = \left\{ f : f \in [0,1], \int_\Omega f dx = C(n)\beta \right\}, \; C(n) \text{ is in}(4.27).$$

Let $\Phi_{s_k} = [u_k]_{s_k}^2$. Let $\hat{f}_k \in \bar{\mathcal{R}}_{\tilde{\beta}}$ be the unique solution to the minimization problem

$$\Phi_{s_k}(\hat{f}_k) = \inf_{f_k \in \bar{\mathcal{R}}_{\tilde{\beta}}} \Phi_{s_k}(f_k).$$

Let $\hat{u}_k = u_{\hat{f}_k}$. Furthermore, we have

$$\Phi(f) = \int_\Omega |\nabla u_f|^2 dx,$$

where in this section, u_f is the solution to

$$\begin{cases} -\Delta u_f = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Also, we denote by $f^* \in \bar{\mathcal{R}}_\beta$ the solution to the minimization problem.

$$\Phi(f^*) = \inf_{f \in \bar{\mathcal{R}}_\beta} \Phi(f).$$

Finally, we denote $u^* = u_{f^*}$. Now, we are ready to introduce the following main result of this section.

Theorem 4.4.1. *Under the above notations, we have the following up to a subsequence when $k \to \infty$,*

1. $\hat{f}_k \overset{*}{\rightharpoonup} C(n) f^*$ *in* $L^\infty(\Omega)$,
2. $\Phi_{s_k}(\hat{f}_k) \to C(n)\Phi(f^*)$,
3. $\hat{u}_k \to u^*$ *in* $L^2(\Omega)$,

where $C(n)$ is in (4.27).

We need the following auxiliary theorem to prove the main result.

Theorem 4.4.2. *Let $s_k \to 1^-$ as $k \to \infty$, and let $f_k, f \in L^2(\Omega)$ be such that $f_k \rightharpoonup f$ weakly in $L^2(\Omega)$. Also, let $u_k \in H_0^{s_k}(\Omega)$ and $u \in W_0^{1,2}(\Omega)$ be the solution to*

$$\begin{cases} (-\Delta)^{s_k} u_k = f_k & \text{in } \Omega, \\ u_k = 0 & \text{in } \mathbb{R}^n \setminus \Omega, \end{cases}$$

and

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

respectively. Then, $u_k \to u$ strongly in $L^2(\Omega)$. Moreover, as $k \to \infty$

$$[u_k]_{s_k}^2 \to C(n)\|\nabla u\|_2^2.$$

where $C(n)$ is in (4.27).

Proof. Let $F_k, F : L^2(\mathbb{R}^n) \to \bar{\mathbb{R}}$ be given respectively by

$$F_k(v) = \begin{cases} \frac{1}{2}[v]_{s_k}^2 - \int_\Omega f_k v dx & \text{if } v \in H_0^{s_k}(\Omega); \\ +\infty & \text{o.w.} \end{cases}$$

and

$$F(v) = \begin{cases} C(n) \cdot \left(\frac{1}{2}\|\nabla v\|_2^2 - \int_\Omega f v dx\right) & \text{if } v \in W_0^{1,2}(\Omega); \\ +\infty & \text{o.w.} \end{cases}$$

We claim that $F = \Gamma - \lim_{k\to\infty} F_k$. Indeed, assume that $\{v_k\} \subset L^2(\mathbb{R}^n)$ is any sequence such that $v_k \to v$ strongly in $L^2(\mathbb{R}^n)$. Then,

$$\lim_{k\to\infty} C(n) \int_\Omega f_k v_k dx = \int_\Omega f v dx.$$

For any $v \in W_0^{1,2}(\Omega)$, Proposition 4.4.2 implies $F(v) \le \liminf_{k\to\infty}[v_k]_{s_k}^2$ and Proposition 4.4.1 implies $F(v) \ge \limsup_{k\to\infty}[v]_{s_k}^2$. For $v \notin W_0^{1,2}(\Omega)$, it is easy to verify that $\liminf_{k\to\infty}[v_k]_{s_k}^2 = \infty$. Thus, the claim is proven. Now, observe that

$$F_k(u_k) = \inf_{v\in L^2(\mathbb{R}^n)} F_k(v) \quad \text{and} \quad F(u) = \inf_{v\in L^2(\mathbb{R}^n)} F(v). \tag{4.38}$$

By Lemma 2.2.3, we have for any $k \in \mathbb{N}$

$$[u_k]_{s_k} \le \|f_k\|_2 \cdot C(n, s_k, \Omega),$$

where $C(n, s_k, \Omega)$ is in (2.9) and is bounded as $k \to \infty$. Hence, $\{u_k\}$ is precompact in $L^2(\Omega)$. Together with (4.38), we conclude that Theorem 2.3.1 implies $u_k \to u$ in $L^2(\Omega)$. Finally, up to a subsequence

$$\lim_{k\to\infty} [u_k]_{s_k}^2 = \lim_{k\to\infty} \int_\Omega f_k u_k dx = C(n) \int_\Omega f u dx = C(n)\|\nabla u\|_2^2,$$

This completes the proof. □

Now, we are ready to prove the main result of this section.

Proof of Theorem 4.4.1. Since $\{\hat{f}_k\} \in \bar{\mathcal{R}}_{\tilde{\beta}}$ is bounded in $L^\infty(\Omega)$, there exists a subsequence (still denoted by $\{\hat{f}_k\}$) and $f' \in \bar{\mathcal{R}}_\beta$ so that

$$\hat{f}_k \overset{*}{\rightharpoonup} C(n)f' \quad \text{in } L^\infty(\Omega).$$

Hence, $\hat{f}_k \rightharpoonup u_{f'}$ in $L^2(\Omega)$. By Theorem 4.4.2, we have that $\hat{u}_k \to u_{f'}$ in $L^2(\Omega)$. Then, Proposition 4.4.2 implies

$$C(n) \inf_{\bar{\mathcal{R}}_\beta} \Phi \le C(n)\Phi(f') = C(n)\|\nabla u_{f'}\|_2^2 \le \liminf_{k\to\infty}[\hat{u}_k]_{s_k}^2 = \liminf_{k\to\infty} \inf_{\bar{\mathcal{R}}_{\tilde{\beta}}} \Phi_{s_k}.$$

Also, Theorem 4.4.2 implies

$$\limsup_{k\to\infty} \inf_{\bar{\mathcal{R}}_{\tilde{\beta}}} \Phi_{s_k} \leq \lim_{k\to\infty} \Phi_{s_k}(f^*) = C(n)\Phi(f^*) = C(n) \inf_{\bar{\mathcal{R}}_{\beta}} \Phi.$$

Consequently, we have $\Phi(f^*) = \Phi(f')$. The uniqueness of optimal load implies $f' = f^*$ *a.e.* and the uniqueness of solution implies $u_{f'} = u^*$. The proof is complete. □

Chapter 5

The Variational Minimization Problem

5.1 Introduction

In this chapter, we are interested in generalizing the results in Section 1.6 to the fractional case. The setting of this variational fractional problem is the following. Let $0 < s < 1$ be fixed, Ω be a bounded open domain in $\mathbb{R}^n$, and f be a non-negative function. To avoid extra notations, in this chapter we will use u_l to denote the solution to

$$\begin{cases} (-\Delta)^s u_l(x) + l u_l(x) = f(x) \text{ in } \Omega, \\ u_l(x) = 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.1}$$

where $l \in \bar{\mathcal{R}}_\beta$ is the so-called *design function.* Also, we define the functional

$$J(l) := [u_l]_s^2 + \int_\Omega l u_l^2 dx = \int_\Omega f u_l dx,$$

where $[\cdot]$ is the Gagliardo-Nirenberg semi-norm (see Definition 2.2.1).

The *weak formulation* of (5.1) is

$$\langle (-\Delta)^s u_l, v \rangle + \int_\Omega l u_l v dx = \int_\Omega f v dx, \tag{5.2}$$

for any $v \in H_0^s(\Omega)$.

Theorem 5.3.1 and Theorem 5.3.2 are the main results which give sufficient and necessary conditions for some $\hat{l}$ in $\bar{\mathcal{R}}_\beta$ to be a minimizer. The technical machinery we will apply is the so-called *tangent cones method.*

5.2 Further Preliminaries

Lemma 5.2.1. *For any $l \in \bar{\mathcal{R}}_\beta$, the equation* (5.1) *has a unique solution u_l which satisfies*

$$\begin{aligned} & -\frac{1}{2}J(l) = -\frac{1}{2}\int_\Omega f u_l dx \\ = & -\frac{1}{2}[u_l]_s^2 - \frac{1}{2}\int_\Omega l {u_l}^2 dx \qquad (5.3) \\ = & \min_{v \in H_0^s(\Omega)} \left(\frac{1}{2}[v]_s^2 + \frac{1}{2}\int_\Omega l v^2 dx - \int_\Omega f v dx \right). \qquad (5.4) \end{aligned}$$

Proof. If u_l is a solution of (5.1), equation (5.3) follows from (5.2). For any $v \in H_0^s(\Omega)$, we take

$$\Psi(v) = \frac{1}{2}[v]_s^2 + \frac{1}{2}\int_\Omega l v^2 dx - \int_\Omega f v dx. \qquad (5.5)$$

Since Ψ is strict convex with respect to v, there is a unique minimizer of Ψ, say u_0. It suffices to show that u_0 is a solution of (5.1) if and only if u_0 minimize Ψ. Take $u_\varepsilon := u_0 + \varepsilon\varphi$ with $\varepsilon \in \mathbb{R}, \varphi \in H_0^s(\Omega)$. Then, $\Psi(u_\varepsilon) \geq \Psi(u_0)$ implies

$$\varepsilon \left(\langle (-\Delta)^s u_0, \varphi \rangle + \int_\Omega l u_0 \varphi dx - \int_\Omega f \varphi \, dx \right) \geq 0.$$

Therefore, we obtain

$$\langle (-\Delta)^s u_0, \varphi \rangle + \int_\Omega l u_0 \varphi dx = \int_\Omega f \varphi dx,$$

as required.

Conversely, assume $u_0 = u_l$ is a solution of (5.1). Then for an arbitrary $u_1 \in H_0^s(\Omega)$, a direct computation gives

$$\begin{aligned} & \Psi(u_1) - \Psi(u_0) \\ = & \frac{1}{2}[u_1]_s^2 + \frac{1}{2}\int_\Omega l u_1^2 dx - \int_\Omega f u_1 dx + \frac{1}{2}[u_l]_s^2 + \frac{1}{2}\int_\Omega l u_l^2 dx \\ = & \frac{1}{2}[u_1]_s^2 + \frac{1}{2}[u_l]_s^2 - \langle (-\Delta)^s u_l, u_1 \rangle + \frac{1}{2}\int_\Omega l u_1^2 dx + \frac{1}{2}\int_\Omega l u_l^2 dx \\ & - \int_\Omega l u_l u_1 dx \\ = & \frac{1}{2}\iint_{\mathbb{R}^{n \times n}} \frac{(u_1(x) - u_l(x) - u_1(y) + u_l(y))^2}{|x-y|^{n+2s}} dxdy + \frac{1}{2}\int_\Omega l(u_1 - u_l)^2 dx \\ & \geq 0. \end{aligned}$$

The proof is complete. □

5.3 Existence and characteristic formula of the solution

Theorem 5.3.1. *J is convex and weak$^*-$continuous in $L^\infty(\Omega)$. In particular, there exists $\hat{l}$ in $\bar{\mathcal{R}}_\beta$ such that*

$$\inf_{\omega\in\mathcal{R}_\beta} J(\omega) = \min_{l\in\bar{\mathcal{R}}_\beta} J(l) = J(\hat{l}). \tag{5.6}$$

Proof. We take $\Psi(v,l)$ as in (5.5). For $v \in H_0^s(\Omega)$ fixed, $\Psi(v,l)$ is affine, and hence concave with respect to l. For $l \in L^\infty(\Omega)$ fixed, $v = u_l$ realises the minimum of $\Psi(v,l)$ over $H_0^s(\Omega)$ (see the proof of Lemma 5.2.1). By weak formulation (5.2), we have

$$-\frac{1}{2}J(l) = \min_{v\in H_0^s(\Omega)} \Psi(v,l).$$

which implies that $-\frac{1}{2}J(l)$ is the pointwise minimum of a collection of affine functions. Therefore, $-\frac{1}{2}J(l)$ is also concave, and hence $J(l)$ is convex in $L^\infty(\Omega)$ as required.

Now, let l_n be a sequence in $L^\infty(\Omega)$ converging to some $l \in \bar{\mathcal{R}}_\beta$ in weak* topology. Let us denote by u_n the solution of (5.1) with design function l_n and by u_l the solution of (5.1) with design function l. In this case, the function $w_n := u_n - u_l$ satisfies

$$\begin{cases} (-\Delta)^s w_n + l_n w_n = (l - l_n)u_l & \text{in } \Omega, \\ w_n = 0 \qquad \text{in } \mathbb{R}^n \setminus \Omega, \end{cases} \tag{5.7}$$

and the weak formulation gives

$$[w_n]_s^2 + \int_\Omega l_n {w_n}^2 dx = \int_\Omega (l - l_n)u_l w_n dx. \tag{5.8}$$

By Lemma 2.2.3, weak formulation (5.2), *Cauchy-Schwartz inequality*, and the fact that l is non-negative, we have that

$$[u_n]_s^2 \le \int_\Omega f u_n dx \le \|f\|_2 \|u_n\|_2 \le C\|f\|_2 [u_n]_s,$$

which implies that $[u_n]_s \le C\|f\|_2$. Therefore, u_n is bounded in $H_0^s(\Omega)$ and thus is also bounded in $L^2(\Omega)$. By Lemma 2.2.4, there exists a subsequence (still denoted by u_n) and a function $v \in L^2(\Omega)$ such that $u_n \to v$ in $L^2(\Omega)$. Take $w^* := v - u_l$ and we get $w_n \to w^*$ in $L^2(\Omega)$. Therefore $u_l w_n \to u_l w^*$ in $L^2(\Omega)$. Since $l_n \stackrel{*}{\rightharpoonup} l$ in $L^2(\Omega)$, the right hand side of (5.8) vanishes as $n \to \infty$. Moreover, the term $\int_\Omega l_n {w_n}^2 dx$ is non-negative. Then w_n converges to 0 in $H_0^s(\Omega)$, and 0 is the only accumulation point of the whole sequence. Thus, J is weak$^*-$continuous in $L^\infty(\Omega)$. Since $J(l)$ is non-negative, there is a minimization subsequence. Therefore the existence of the minimizer follows from Banach-Alaogu's theorem. Finally, the first inequality in (5.6) is the consequence of the fact that $\bar{\mathcal{R}}_\beta$ is the weak* closed hull of $\mathcal{R}_\beta$. □

In this case, we are going to study the optimality conditions of the minimizer $\hat{l}$. Note that in the following lemma, we have $l \in L^\infty(\Omega)$ instead of $l \in \bar{\mathcal{R}}_\beta$. Hence this lemma is essential in this section.

Lemma 5.3.1. *For any $l \in L^\infty(\Omega)$, $J(l)$ is well-defined.*

Proof. Let u_1 and u_2 solve (5.1) with $u_1 \neq u_2$. We need to show that

$$\int_\Omega f u_1 dx = \int_\Omega f u_2 dx.$$

Observe that we have the equations

$$\begin{cases} (-\Delta)^s(u_1 - u_2) + l(u_1 - u_2) = 0 \text{ in } \Omega \\ u_1 - u_2 = 0 \quad\quad\quad\quad\quad\quad\quad \text{in } \mathbb{R}^n \setminus \Omega. \end{cases} \tag{5.9}$$

Taking $v = u_1 + u_2$ as the test function in (5.9), we have

$$\begin{aligned} & \langle (-\Delta)^s(u_1 - u_2), u_1 + u_2 \rangle + \langle l(u_1 - u_2), u_1 + u_2 \rangle \\ = & \iint_{\mathbb{R}^{n\times n}} \frac{(u_1(x) - u_1(y)) - (u_2(x) - u_2(y))}{|x-y|^{n+2s}} \\ & \cdot \left((u_1(x) - u_1(y)) + (u_2(x) - u_2(y)) \right) dxdy \\ + & \int_\Omega l(x) \left(u_1^2(x) - u_2^2(x) \right) dx = 0, \end{aligned}$$

or

$$[u_1]_s^2 + \int_\Omega l u_1^2 dx = [u_2]_s^2 + \int_\Omega l u_2^2 dx,$$

as required. □

Lemma 5.3.2. *The functional J is Frechet-differentiable at any point $l \in \bar{\mathcal{R}}_\beta$ and*

$$\langle J'(l), h \rangle = \int_\Omega -h u_l^2 dx,$$

where u_l is the solution of (5.1).

Proof. Fix h in $L^\infty(\Omega)$. Subtracting equation (5.1) with design function $l + h$ in the one with design function l, we have

$$(-\Delta)^s(u_{l+h} - u_l) + l(u_{l+h} - u_l) = -h u_{l+h} \quad \text{in } \Omega. \tag{5.10}$$

Taking $v = u_{l+h} + u_l$ as the test function

$$\begin{aligned} & \iint_{\mathbb{R}^{n\times n}} \frac{(u_{l+h}(x) - u_{l+h}(y)) - (u_l(x) - u_l(y))}{|x-y|^{n+2s}} \\ & \cdot \left((u_{l+h}(x) - u_{l+h}(y)) + (u_l(x) - u_l(y))\right) dxdy \\ & + \int_\Omega l u_{l+h}^2(x) - l u_l^2(x) dx \\ = & -\int_\Omega h u_{l+h} \left(u_{l+h}(x) + u_l(x)\right) dx, \end{aligned}$$

or

$$\begin{aligned} & [u_{l+h}]_s^2 - [u_l]_s^2 + \int_\Omega l u_{l+h}^2 - l u_l^2 dx \\ = & -\int_\Omega h u_{l+h} \left(u_{l+h} + u_l\right) dx, \end{aligned}$$

or

$$J(l+h) - J(l) + \int_\Omega h u_l^2 dx = -\int_\Omega h u_l \left(u_{l+h} - u_l\right) dx,$$

and by *Cauchy-Schwartz inequality*

$$\left| J(l+h) - J(l) + \int_\Omega h u_l^2 dx \right| \leq \|h\|_\infty \|u_{l+h} - u_l\|_2 \|u_l\|_2. \tag{5.11}$$

Again, we take $v = u_{l+h} - u_l$ as the test function in (5.10) and use Lemma 2.2.3. Since l is non-negative, we obtain

$$\begin{aligned} & \frac{1}{C}\|u_{l+h} - u_l\|_2^2 \leq [u_{l+h} - u_l]_s^2 + \int_\Omega l \left(u_{l+h} - u_l\right)^2 dx \\ = & -\int_\Omega h u_{l+h}(u_{l+h} - u_l) dx, \end{aligned}$$

or

$$\|u_{l+h} - u_l\|_2 \leq C\|h u_{l+h}\|_2.$$

As in the proof of Theorem 5.3.1, we have

$$\begin{aligned} & \|u_{l+h} - u_l\|_2 \leq C\|h\|_\infty \|u_l + h\|_2 \\ & \leq C^{3/2}\|h\|_\infty [u_{l+h}]_s \leq C^{5/2}\|h\|_\infty \|f\|_2. \end{aligned}$$

Therefore, the right hand side of (5.11) converges to 0 uniformly as $\|h\|_\infty \to 0$, as required. □

We refer to [5] and [45] for the concept of *tangent cone* as following.

Definition 5.3.1. *For any subset A of a Banach space Y, and any $a \in A$, we denote the tangent cone of A at a by $T'_A(a)$. We say $v \in T'_A(a)$ if and only if for each $t_n \to 0^+$ there exists a sequence $v_n \in Y$ satisfying*

1. $\lim_{n\to\infty} \|v_n - v\|_Y = 0$,

2. for each n*, we have* $a + t_n v_n \in A$.

In this thesis, we denote the tangent cone of the set $\bar{\mathcal{R}}_\beta$ at the point $l \in \bar{\mathcal{R}}_\beta$ by $T'(l)$. By [37], we have the following lemma.

Lemma 5.3.3. *The tangent cone* $T'(l)$ *is the set of all* $h \in L^\infty(\Omega)$ *such that*

1.
$$\int_\Omega h(x)dx = 0,$$

2.
$$\left\|\chi_{Q_n^0} h^-\right\|_\infty \to 0,$$

3.
$$\left\|\chi_{Q_n^1} h^+\right\|_\infty \to 0,$$

where
$$Q_n^0 = \left\{x \in \Omega, l(x) \leq \frac{1}{n}\right\},$$

and
$$Q_n^1 = \left\{x \in \Omega, l(x) \geq 1 - \frac{1}{n}\right\}.$$

Remark 5.3.1. $\hat{l} \in \bar{\mathcal{R}}_\beta$ *is the minimizer of* J *if and only if*

$$\forall h \in T'(\hat{l}), \left\langle J'(\hat{l}), h\right\rangle \geq 0.$$

Proof. For any $h \in T'(\hat{l})$, Lemma 5.3.3 implies that $\hat{l} + \varepsilon h \in \bar{\mathcal{R}}_\beta$ as $\varepsilon \to 0^+$. By Lemma 5.3.2,

$$J(\hat{l} + \varepsilon h) - J(\hat{l}) = -\varepsilon \int_\Omega h u_{\hat{l}}^2 dx + o(\varepsilon^2).$$

By minimality of $\hat{l}$, we have $\left\langle J'(\hat{l}), h\right\rangle \geq 0$. In contrast, assume $\hat{l}$ is not the minimizer. Then, there exists some $l^* \in \bar{\mathcal{R}}_\beta$ such that $J(l^*) < J(\hat{l})$. Take $h := l^* - \hat{l}$. It is easy to see $h \in T'(\hat{l})$. Then take $g(\varepsilon) := J(\hat{l} + \varepsilon h) - J(\hat{l})$ with $\varepsilon \in [0, 1]$. Since J is convex, $g(\varepsilon)$ is also convex, and

$$0 \leq \left\langle J'(\hat{l}), h\right\rangle = g'(0) \leq \frac{g(1) - g(0)}{1 - 0} < 0,$$

which is a contradiction. □

The following lemma shows that u is non-negative *a.e.*.

Lemma 5.3.4. *Let l be in $\bar{\mathcal{R}}_\beta$ and let u_l solves* (5.1). *Then* $|\{u_l < 0\}| = 0$.

Proof. Take $\omega = \text{supp } u_l^-$. Let us assume $|\omega| > 0$. Then, let us take test function $v = u_l^-$ in the weak form (5.2). We obtain

$$\langle(-\Delta)^s u_l, v\rangle = -\int_\Omega l u_l u_l^- \,dx + \int_\Omega f u_l^- /, dx \tag{5.12}$$

$$= \int_\omega l(u_l^-)^2 \,dx + \int_\omega f u_l^- \,dx \geq 0.$$

However, the left hand side of (5.12) gives

$$\left\langle(-\Delta)^s u_l, u_l^-\right\rangle = \iint_{\mathbb{R}^{n\times n}} \frac{(u_l(x) - u_l(y)) \cdot \left(u_l^-(x) - u_l^-(y)\right)}{|x-y|^{n+2s}} \,dxdy$$

$$= -\int_\omega \int_\omega \frac{\left(u_l^-(x) - u_l^-(y)\right)^2}{|x-y|^{n+2s}} \,dxdy$$

$$+ \int_{\mathbb{R}^n\setminus\omega} \int_\omega \frac{-\left(u_l^-(x)\right)^2 - u_l^-(x)u_l(y)}{|x-y|^{n+2s}} \,dxdy$$

$$+ \int_\omega \int_{\mathbb{R}^n\setminus\omega} \frac{-\left(u_l^-(y)\right)^2 - u_l^-(y)u_l(x)}{|x-y|^{n_+2s}} \,dxdy < 0,$$

which is a contradiction. □

For any $l \in \bar{\mathcal{R}}_\beta$, we denote the following sets up to a set of zero measure,

$$\begin{cases} \Omega_0 = \{x \in \Omega, l(x) = 0\}, \\ \Omega_1 = \{x \in \Omega, l(x) = 1\}, \\ \Omega_* = \{x \in \Omega, l(x) \in (0,1)\}. \end{cases} \tag{5.13}$$

Now, we have the following main result.

Theorem 5.3.2. *Let $\hat{l}$ be in $\bar{\mathcal{R}}_\beta$ and $\hat{u}$ solves* (5.1). *We use the notations in* (5.13). *Then $\hat{l}$ minimizes J if and only if the following two conditions holds*

1. *If $|\Omega_*| > 0$, $\hat{u}$ is constant on Ω_*.*
2. *For any $x_1 \in \Omega_1$, $x_* \in \Omega_*$ and $x_0 \in \Omega_0$, we have*

$$\hat{u}(x_0) \leq \hat{u}(x_*) \leq \hat{u}(x_1).$$

Proof. Thanks to Lemma 5.3.4, we prove the theorem as follows. Let $\hat{l}$ be the minimizer of J. We take the increasing union $\Omega_* = \bigcup_{n>1} \Omega_*^n$ where

$$\Omega_*^n = \left\{x \in \Omega, \hat{l}(x) \in \left(\frac{1}{n}, 1 - \frac{1}{n}\right)\right\}.$$

It suffices to prove that $\hat{u}$ is constant on Ω_*^n for every $n > 2$. Assume, for some n, that $\hat{u}$ is not constant on Ω_*^n. Then it is possible to find two subsets A and B with positive measure such that $|A| = |B|$ and

$$\int_A \hat{u}^2 dx < \int_B \hat{u}^2 dx. \tag{5.14}$$

Then, we take

$$h(x) = \begin{cases} -1 \text{ in } A, \\ 1 \quad \text{in } B, \\ 0 \quad \text{otherwise.} \end{cases} \tag{5.15}$$

By Lemma 5.3.3, $h \in T'(\hat{l})$ and we use (5.14) to obtain,

$$\left\langle T'(\hat{l}), h \right\rangle = -\int_\Omega h\hat{u}^2 dx = \int_A \hat{u}^2 dx - \int_B \hat{u}^2 dx < 0,$$

which contradicts Remark 5.3.1, and we have proven that $\hat{u}$ is constant in Ω_*.

Now, assume the contrary and thus there exists a subset B with sufficiently small positive measure in Ω_0 such that

$$\hat{u}|_B > \hat{u}|_{\Omega_*} = \text{constant}.$$

Then, we find A contained in Ω_*^n for some n such that $|A| = |B|$. By selecting h in (5.15), we reach a contradiction as above. Similarly, we prove $\hat{u}|_{\Omega_1} \geq \hat{u}|_{\Omega_*}$.

Conversely, assume that $(\bar{u}, \bar{l})$ satisfies condition (1) and (2) (with constant C). Take any $h \in T'(\bar{l})$. Obviously, h is non-negative in Ω_0 and non-positive in Ω_1. Then we obtain

$$\begin{aligned} & \left\langle T'(\bar{l}), h \right\rangle = -\int_\Omega h\bar{u}^2 dx \\ = \ & -\int_{\Omega_0} h\bar{u}^2 dx - \int_{\Omega_*} h\bar{u}^2 dx - \int_{\Omega_1} h\bar{u}^2 dx \\ \geq \ & -\int_{\Omega_0} hC^2 dx - \int_{\Omega_*} hC^2 dx - \int_{\Omega_1} hC^2 dx \\ = \ & -C^2 \int_\Omega h dx = 0. \end{aligned}$$

By remark 5.3.1, $\bar{l}$ is the minimizer of J. $\square$

Chapter 6

Conclusion and Open Areas

6.1 Conclusion

This is our first work in the direction of optimal rearrangement problems for fractional equations. It has established a solid base for future research in the area. In particular optimal rearrangement problems for the $p-$fractional Laplacian, as well as constrained and more complicated variational problems are of great interest.

One of the major problems in optimal rearrangement is the question whether or not the solution to the relaxed problem, i.e., maximization / minimization over $\bar{\mathcal{R}}_\beta$, is also the solution to the original problem, or, is the original problem solvable or not.

The fact that the fractional minimizer is not a bang-bang function ($\hat{f} \in \bar{\mathcal{R}}_\beta \setminus \mathcal{R}_\beta$) is also an important result showing that the classical case is rather an exception. A similar result has been obtained recently in [44].

Further we believe the derived fractional normalized obstacle equation (4.16) is of great interest especially for the development of fast converging numerical algorithms.

The maximization problem, motived by [2], is also an important result, showing that fractional maximizers are bang-bang solutions, and that the optimal rearrangement problem is solvable.

For the variational minimization problem related to semi-linear PDE considered in Chapter 5, the answer is in general negative also in the classical case. Further research in this direction is of great interest.

6.2 Open Questions

There are two open questions in the context of this thesis arising from Chapter 3 and Chapter 5, which we would like to work in future.

In Chapter 3, for any minimizer $\hat{f}$ and the corresponding solution $\hat{u}$, the open question is to show whether the flat part $\{\hat{u} = \alpha\}$ has zero measure (see Figure 3.1). Our efforts in this direction have not been successful so far.

In Chapter 5, the open question is to show whether we could generalize Theorem 1.6.2 in fractional setting. To explain it, the task is to find some conditions on β and f so that the minimizer $\hat{l}$ is a characteristic function, or equivalently in $\mathcal{R}_\beta$. The main difficulty arises from the non-locality of the fractional Laplace operators, but one may be interested in conditions on data with which the minimizer is **never** a characteristic function. The methods in [37] heavily rely on the locality of the Laplace operators, and are not applicable in non-local setting.

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